

ON WEAK SOLUTIONS OF SDES WITH SINGULAR TIME-DEPENDENT DRIFT AND DRIVEN BY STABLE PROCESSES

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ABSTRACT. Let $d \geq 2$. In this paper, we study weak solutions for the following type of stochastic differential equation

$$\begin{cases} dX_t = dS_t + b(s+t, X_t)dt, & t \geq 0, \\ X_0 = x, \end{cases}$$

where $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ is the initial starting point, $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable, and $S = (S_t)_{t \geq 0}$ is a d -dimensional α -stable process with index $\alpha \in (1, 2)$. We show that if the α -stable process S is non-degenerate and $b \in L_{loc}^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d)) + L_{loc}^q(\mathbb{R}_+; L^p(\mathbb{R}^d))$ for some $p, q > 0$ with $d/p + \alpha/q < \alpha - 1$, then the above SDE has a unique weak solution for every starting point $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

1. Introduction

In this paper, we study the following type of stochastic differential equation

$$\begin{cases} dX_t = dS_t + b(s+t, X_t)dt, & t \geq 0, \\ X_0 = x, \end{cases} \quad (1.1)$$

where $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ is the initial starting point, $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable, and $S = (S_t)_{t \geq 0}$ is a d -dimensional α -stable process. The equation (1.1) is a shorthand way of writing

$$X_t = x + S_t + \int_0^t b(s+u, X_u)du, \quad t \geq 0. \quad (1.2)$$

Since the drift b is not assumed to be bounded, solutions of (1.1) are supposed to fulfill the integrability conditions

$$\int_0^t |b(s+u, X_u)|du < \infty \quad \text{a.s.,} \quad \forall t \geq 0, \quad (1.3)$$

such that the integral in (1.2) makes sense.

The classical results tell us, that if b is of linear growth and globally Lipschitz in the space variable, then a unique strong solution to (1.1) exists. However, it turns out that the Lipschitz-continuity on the drift is far from being necessary; this is well demonstrated in the special case where $\alpha = 2$, that is, when S is a

2000 *Mathematics Subject Classification.* primary 60H10, 60J75; secondary 60J35.

Key words and phrases. stochastic differential equations, singular drift, stable process, weak solutions, martingale problem, resolvent.

d -dimensional Brownian motion. Indeed, there is an extensive literature devoted to the study of Brownian motion (or more generally, diffusions) with singular drift, see, e.g., [26, 22, 16, 21, 1, 13, 6, 24], and many others. In particular, it was shown in [13] that if S is a Brownian motion and there exist $p, q > 0$ with $d/p + 2/q < 1$ such that $b \in L^q_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^d))$, namely

$$\int_0^T \left(\int_{\mathbb{R}^d} |b(t, x)|^p dx \right)^{q/p} dt < \infty, \quad \forall T > 0, \quad (1.4)$$

then strong existence and uniqueness hold for (1.1). Regarding weak solutions to (1.1) in the Brownian case, the condition (1.4) can be relaxed, see, e.g., [1, 9], where weak existence and uniqueness for (1.1) (in the case $\alpha = 2$) were shown for some Kato-class drifts.

There is now a growing interest to study (1.1) for the case where $\alpha \in (0, 2)$. Earlier works in this direction include [12, 15], which primarily concentrated on weak solutions, or equivalently, solutions to the corresponding martingale-problem. Recently, weak existence and uniqueness of rotationally symmetric α -stable ($1 < \alpha < 2$) processes with (time-independent) singular drift belonging to the Kato-class $\mathcal{K}_{d, \alpha-1}$ were obtained in [3, 11]. Compared to weak solutions, one needs generally more regularity on the drift to obtain strong solutions to (1.1), as seen in the diffusion case; this is also the case when $\alpha \in (0, 2)$. Priola [17] proved that if the stable process S is symmetric, non-degenerate and with index $\alpha \in (1, 2)$, and b is time-independent and belongs to $C_b^\beta(\mathbb{R}^d)$ with $\beta > 1 - \alpha/2$, then pathwise uniqueness holds for (1.1). Afterwards, similar results were obtained by Zhang [25] where it was shown that if S is as in [17], b is locally bounded and there exist some $\beta \in (1 - \alpha/2, 1)$ and $p > 2d/\alpha$ such that for any $T, R > 0$,

$$\sup_{t \in [0, T]} \int_{\{|x| \leq R\}} \int_{\{|y| \leq R\}} \frac{|b(t, x) - b(t, y)|^p}{|x - y|^{d+\beta p}} dx dy < \infty,$$

then a unique strong solution to (1.1) exists. As an intermediate step to obtain the main result, Zhang [25] also obtained the following result on weak existence: if the stable process S is symmetric, non-degenerate and with index $\alpha \in (1, 2)$, and there exist $p, q > 0$ such that

$$d/p + \alpha/q < \alpha - 1 \quad \text{and} \quad b \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^d)) + L^q_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^d)), \quad (1.5)$$

then weak solutions to (1.1) exist. Very recently, the results of [17] have been extended in [2] to a larger class of Lévy processes, including the rotationally symmetric α -stable process with index $0 < \alpha \leq 1$. For SDEs with irregular drift and driven by other types of Lévy noises, see also [7, 14].

In this paper, we study weak solutions to (1.1) for the case $1 < \alpha < 2$. We are mainly interested in the weak uniqueness problem. Under mild assumptions on the stable process, we will prove that the condition (1.5) on b implies weak uniqueness for (1.1). More precisely, the main result of the present paper is as follows:

Theorem 1.1. *Let $d \geq 2$ and $1 < \alpha < 2$. Assume that the α -stable process S is non-degenerate, that is, Assumption 2.1 (see Section 2) is satisfied. Assume that b*

is such that (1.5) holds. Then the SDE (1.1) has a unique weak solution for every starting point $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

We remark that the stable process S considered in Theorem 1.1 is not necessarily symmetric. Therefore, Theorem 1.1 also extends the above mentioned result of [25] on weak existence for (1.1).

We now briefly discuss our strategy to prove Theorem 1.1. Essentially, our proof of Theorem 1.1 is based on some perturbation arguments. Under the assumptions of Theorem 1.1, we will see that the time-space resolvent of the solution X to (1.1) can be explicitly expressed in terms of a series, of which the main term is the time-space resolvent of S . This enables us to obtain weak uniqueness for (1.1). On the other hand, some estimates on the time-space resolvent of X can be established and used as substitutes of the Krylov-estimates obtained in [25]; as a consequence, we can adapt the proof of weak existence in [25] to make it work also for our case. We would like to point out that the perturbation on the resolvent of S that we will do in this paper depends mainly on the scaling property of the heat kernel of S , rather than exact estimates of that. As shown in [23], sharp heat kernel estimates are actually not available in our case, since S is merely assumed to be non-degenerate. Therefore, we can not carry out the same perturbation on the heat kernel of S as done in [3, 11], where a rotationally symmetric α -stable process is considered for which sharp heat kernel estimates are known.

The rest of the paper is organized as follows. In Section 2 we recall some basic facts on α -stable processes and the definition of the martingale problem for non-local generators. In Section 3 we establish some estimates on the time-space resolvent of non-degenerate α -stable processes and obtain a solvability result on the corresponding resolvent equation. In Section 4 we prove the local existence and uniqueness of weak solutions to (1.1), under slightly stronger assumptions than those in Theorem 1.1. Finally, we prove Theorem 1.1 in Section 5.

2. Preliminaries

Throughout this paper, we assume that dimension $d \geq 2$. The inner product of x and y in \mathbb{R}^d is written as $x \cdot y$. We use $|v|$ to denote the Euclidean norm of a vector $v \in \mathbb{R}^m$, $m \in \mathbb{N}$. For a bounded function $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^m$ we write $\|g\| := \sup_{(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d} |g(s,x)|$. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ be the unitary sphere.

Let $\alpha \in (1, 2)$ be fixed throughout this paper. A d -dimensional α -stable process $S = (S_t)_{t \geq 0}$ is a Lévy process with a particular form of characteristic exponent ψ , namely

$$\begin{aligned} \mathbf{E}[e^{iS_t \cdot u}] &= e^{-t\psi(u)}, \quad u \in \mathbb{R}^d, \\ \psi(u) &= - \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{iu \cdot y} - 1 - iu \cdot y \right) \nu(dy) - iu \cdot \gamma, \end{aligned} \quad (2.1)$$

where $\gamma \in \mathbb{R}^d$ and the Lévy measure ν is given by

$$\nu(B) = \int_{\mathbb{S}^{d-1}} \mu(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Here, γ is called the center of S and $\gamma = \mathbf{E}[S_1]$; the measure μ is a finite measure on the unitary sphere \mathbb{S}^{d-1} and is called the spectral measure of the α -stable process S . It's worth noting that the first term on the right-hand side of (2.1) is a homogeneous function (with variable u) of index α . As a consequence,

$$\psi(\rho u) + i(\rho u \cdot \gamma) = \rho^\alpha(\psi(u) + i(u \cdot \gamma)), \quad \forall \rho > 0. \quad (2.2)$$

Throughout this paper, we assume S to be non-degenerate, that is, the spectral measure μ of S satisfies the following condition.

Assumption 2.1. The support of μ is not contained in a proper linear subspace of \mathbb{R}^d .

The infinitesimal generator A of the process S is given by

$$Af(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+z) - f(x) - z \cdot \nabla f(x)) \nu(dz) + \sum_{i=1}^d \gamma_i \partial_{x_i} f(x), \quad f \in C_b^2(\mathbb{R}^d), \quad (2.3)$$

where $C_b^2(\mathbb{R}^d)$ denotes the class of C^2 functions such that the function and its first and second order partial derivatives are bounded. Note that $C_b^2(\mathbb{R}^d)$ is a Banach space endowed with the norm

$$\|f\|_{C_b^2(\mathbb{R}^d)} := \|f\| + \sum_{i=1}^d \|\partial_i f\| + \sum_{i,j=1}^d \|\partial_{ij}^2 f\|, \quad f \in C_b^2(\mathbb{R}^d),$$

where $\partial_i f(x) := \partial_{x_i} f(x)$ and $\partial_{ij}^2 f(x) := \partial_{x_i x_j}^2 f(x)$ for $x \in \mathbb{R}^d$. For a function $f(x, y)$ with two variables $x, y \in \mathbb{R}^d$, we also use the notation $A_x f(x, y)$ to indicate that A is operating on the function $f(\cdot, y)$ with y being considered as fixed.

Let

$$L_t := A + b(t, \cdot) \cdot \nabla, \quad (2.4)$$

where ∇ is the gradient operator with respect to the spatial variable.

Let $D = D([0, \infty))$, the set of paths that are right continuous with left limits, endowed with the Skorokhod topology. Set $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and let $\mathcal{D} = \sigma(X_t : 0 \leq t < \infty)$ and $\mathcal{F}_t := \sigma(X_r : 0 \leq r \leq t)$. A probability measure \mathbf{P} on (D, \mathcal{D}) is called a solution to the martingale problem for L_t starting from (s, x) , if

$$\mathbf{P}(X_t = x, \forall t \leq s) = 1, \quad \mathbf{P}\left(\int_s^t |b(u, X_u)| du < \infty, \forall t \geq s\right) = 1, \quad (2.5)$$

and under the measure \mathbf{P} ,

$$f(X_t) - \int_s^t L_u f(X_u) du \quad (2.6)$$

is an \mathcal{F}_t -martingale after time s for all $f \in C_b^2(\mathbb{R}^d)$.

3. Some Analytical Results

We first recall that ψ is the characteristic exponent of the stable process $(S_t)_{t \geq 0}$. According to Assumption 2.1 and [18, Prop. 24.20], there exists some constant $c > 0$ such that

$$|\mathbf{E}[e^{iS_t \cdot u}]| = |e^{-t\psi(u)}| \leq e^{-ct|u|^\alpha}, \quad u \in \mathbb{R}^d. \quad (3.1)$$

By the inversion formula of Fourier transform, the law of S_t has a density $p_t \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ that is given by

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-t\psi(u)} du, \quad x \in \mathbb{R}^d, \quad t > 0. \quad (3.2)$$

According to [23, p. 2856, (2.3)], we have the following scaling property for p_t :

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x + (1 - t^{1-1/\alpha})\gamma), \quad x \in \mathbb{R}^d, \quad t > 0. \quad (3.3)$$

The following result is a slight extension of [17, Lemma 3.1].

Lemma 3.1. *Let $p \geq 1$ be arbitrary. Then for each $t > 0$, the density p_t of S_t and all its derivatives $D^k p_t$ belong to $C_b^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $k = (k_1, \dots, k_d)$ is a multi-index with $k_i \in \mathbb{Z}_+$, $i = 1, \dots, d$, and*

$$D^k := \frac{\partial^{|k|}}{(\partial x_1)^{k_1} \dots (\partial x_d)^{k_d}} \quad \text{with} \quad |k| = k_1 + \dots + k_d.$$

Proof. We follow the proof of [17, Lemma 3.1]. We will only prove the assertion for p_t , since the cases for the derivatives $D^k p_t$ are similar. By the scaling property (3.3), it suffices to consider $t = 1$. As shown in the proof of [17, Lemma 3.1], the characteristic exponent ψ can be written as the sum of ψ_1 and ψ_2 , where

$$\psi_1(u) = - \int_{\{0 < |y| \leq 1\}} \left(e^{iu \cdot y} - 1 - iu \cdot y \right) \nu(dy), \quad \psi_2 = \psi - \psi_1.$$

It is easy to see that $\exp(-\psi_2)$ is bounded and is the characteristic function of an infinitely divisible probability measure m on \mathbb{R}^d . It follows from (3.1) that

$$|e^{-\psi_1(u)}| \leq c_1 e^{-c|u|^\alpha}, \quad u \in \mathbb{R}^d,$$

for some constant $c_1 > 0$. We can easily check that $\psi_1 \in C^\infty(\mathbb{R}^d)$ and that $\exp(-\psi_1)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Since the Fourier transform is a one-to-one map of $\mathcal{S}(\mathbb{R}^d)$ onto itself, we can find $f \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{f} = \exp(-\psi_1)$, where \hat{f} denotes the Fourier transform of f . In particular, we have $f \in L^p(\mathbb{R}^d)$ for all $p \geq 1$. Let $f * m$ be the convolution of f and γ . We have

$$\widehat{f * m} = \hat{f} \hat{m} = e^{-\psi_1 - \psi_2} = e^{-\psi} = \hat{p}_1,$$

which implies $p_1 = f * m$. Thus $p_1 \in C_b^\infty(\mathbb{R}^d)$. By Young's inequality, we get $p_1 \in L^p(\mathbb{R}^d)$. \square

Recall that the operator A defined in (2.3) is the infinitesimal generator of the process S . The following corollary is straightforward.

Corollary 3.2. *For each $t > 0$, we have*

$$Ap_t(x) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(-u) e^{-t\psi(u)} e^{-iu \cdot x} du, \quad x \in \mathbb{R}^d. \quad (3.4)$$

Proof. According to Lemma 3.1, $p_t \in C_b^\infty(\mathbb{R}^d)$. Thus Ap_t is well-defined. For $x \in \mathbb{R}^d$, we have

$$\begin{aligned} Ap_t(x) &= \int_{\mathbb{R}^d \setminus \{0\}} (p_t(x+z) - p_t(x) - z \cdot \nabla p_t(x)) \nu(dz) + \sum_{i=1}^d \gamma_i \partial_{x_i} p_t(x) \\ &= \int_{\mathbb{R}^d \setminus \{0\}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (e^{-iu \cdot (x+z)} - e^{-iu \cdot x} + z \cdot iue^{-iu \cdot x}) e^{-t\psi(u)} du \nu(dz) \\ &\quad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (-iu \cdot \gamma) e^{-iu \cdot x} e^{-t\psi(u)} du. \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned} Ap_t(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d \setminus \{0\}} (e^{-iu \cdot (x+z)} - e^{-iu \cdot x} + z \cdot iue^{-iu \cdot x}) e^{-t\psi(u)} \nu(dz) \right) du \\ &\quad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (-iu \cdot \gamma) e^{-iu \cdot x} e^{-t\psi(u)} du \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-t\psi(u)} \int_{\mathbb{R}^d \setminus \{0\}} (e^{-iu \cdot z} - 1 - iz \cdot (-u)) \nu(dz) du \\ &\quad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (-iu \cdot \gamma) e^{-iu \cdot x} e^{-t\psi(u)} du \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot x} (-\psi(-u)) e^{-t\psi(u)} du. \end{aligned}$$

□

Remark 3.3. Let $y \in \mathbb{R}^d$ be fixed. Later in the proof of Proposition 3.8 we will need to calculate $A(p_t(y - \cdot))$. Proceeding as in the proof of Corollary 3.2, we can easily verify that

$$A_x((p_t(y - x))) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(u) e^{-t\psi(u)} e^{-iu \cdot (y-x)} du, \quad x \in \mathbb{R}^d. \quad (3.5)$$

Next, we show that the the right-hand side of (3.5) is an integrable function with respect to the variable y .

Lemma 3.4. *Denote by $g(t, x, y)$ the right-hand side of (3.5), namely,*

$$g(t, x, y) := -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(u) e^{-t\psi(u)} e^{-iu \cdot (y-x)} du, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

Then $\|g(t, x, \cdot)\|_{L^1(\mathbb{R}^d)}$ is finite and uniformly bounded for $(t, x) \in [\delta, \infty) \times \mathbb{R}^d$, where $\delta > 0$ is an arbitrary constant.

Proof. Using (2.2) and a change of variables $u = t^{-1/\alpha}u'$ for the integral in the definition of $g(t, x, y)$, we obtain

$$\begin{aligned}
g(t, x, y) &= -\frac{t^{-d/\alpha}}{(2\pi)^d} \int_{\mathbb{R}^d} (t^{-1}(\psi(u') + iu' \cdot \gamma) - it^{-1/\alpha}u' \cdot \gamma) \\
&\quad \times e^{-(\psi(u') + iu' \cdot \gamma) + it^{1-1/\alpha}u' \cdot \gamma} e^{-it^{-1/\alpha}u' \cdot (y-x)} du' \\
&= -\frac{t^{-d/\alpha}}{(2\pi)^d} \int_{\mathbb{R}^d} (t^{-1}\psi(u') + iu' \cdot \gamma(t^{-1} - t^{-1/\alpha})) \\
&\quad \times e^{-\psi(u') + iu' \cdot \gamma(t^{1-1/\alpha} - 1)} e^{-it^{-1/\alpha}u' \cdot (y-x)} du' \\
&= -\frac{t^{-1-d/\alpha}}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(u') e^{-\psi(u')} e^{-iu' \cdot t^{-1/\alpha}(y-x-\gamma t + \gamma t^{1/\alpha})} du' \\
&\quad - i \frac{t^{-1-d/\alpha} - t^{-(d+1)/\alpha}}{(2\pi)^d} \int_{\mathbb{R}^d} \gamma \cdot u' e^{-\psi(u')} e^{-iu' \cdot t^{-1/\alpha}(y-x-\gamma t + \gamma t^{1/\alpha})} du'.
\end{aligned}$$

With another change of variables $y' = t^{-1/\alpha}(y - x - \gamma t + \gamma t^{1/\alpha})$, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} |g(t, x, y)| dy &\leq \frac{t^{-1}}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \psi(u') e^{-\psi(u')} e^{-iu' \cdot y'} du' \right| dy' \\
&\quad + \frac{|t^{-1} - t^{-1/\alpha}|}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \gamma \cdot u' e^{-\psi(u')} e^{-iu' \cdot y'} du' \right| dy'. \quad (3.6)
\end{aligned}$$

Let ψ_1 and ψ_2 be as in the proof of Lemma 3.1. We can further write $\psi_2 = \psi_{21} + \psi_{22}$, where

$$\psi_{21}(u) = - \int_{\{|y|>1\}} e^{iu \cdot y} \nu(dy)$$

and

$$\psi_{22}(u) = \int_{\{|y|>1\}} (1 + iu \cdot y) \nu(dy) - iu \cdot \gamma.$$

Then

$$\begin{aligned}
\psi e^{-\psi} &= \psi_1 e^{-\psi_1} e^{-\psi_2} + \psi_2 e^{-\psi_1} e^{-\psi_2} \\
&= \psi_1 e^{-\psi_1} e^{-\psi_2} - e^{-\psi_1} (-\psi_{21}) e^{-\psi_2} + \psi_{22} e^{-\psi_1} e^{-\psi_2}.
\end{aligned}$$

As shown in the proof of Lemma 3.1, $\exp(-\psi_1) \in \mathcal{S}(\mathbb{R}^d)$; similarly, we have $\psi_1 \exp(-\psi_1), \psi_{22} \exp(-\psi_1) \in \mathcal{S}(\mathbb{R}^d)$. Noting that $-\psi_{21}$ and $e^{-\psi_2}$ are both characteristic functions of some finite measures on \mathbb{R}^d , we can argue as in the proof of Lemma 3.1 to conclude that

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \psi(u') e^{-\psi(u')} e^{-iu' \cdot y'} du' \right| dy' < \infty.$$

The finiteness of the integral appearing in the second term on the right-hand side of (3.6) can be similarly proved. Now, the assertion follows from (3.6). \square

The following two lemmas will be used to obtain a solvability result about the parabolic resolvent equation for A ; however, they are interesting in their own right.

Lemma 3.5. *Let (E, \mathcal{M}, m) be a measure space and $f : \mathbb{R}^d \times E \rightarrow \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{M}$ -measurable. Denote by $L^1(E, \mathcal{M}, m)$ the space of all \mathcal{M} -measurable functions on E that are integrable with respect to the measure m . Suppose f satisfies:*

(i) For each $y \in E$, $f(\cdot, y) \in C_b^2(\mathbb{R}^d)$. Moreover, there exist $g_0, g_1, g_2 \in L^1(E, \mathcal{M}, m)$ such that $|f(x, \cdot)| \leq g_0$, $|\nabla_x f(x, \cdot)| \leq g_1$ and $\sum_{i,j=1}^d |\partial_{x_i x_j}^2 f(x, \cdot)| \leq g_2$ for all $x \in \mathbb{R}^d$.

(ii) For each $x \in \mathbb{R}^d$, $f(x, \cdot) \in L^1(E, \mathcal{M}, m)$.

Then

$$A\left(\int_E f(x, y)m(dy)\right) = \int_E A_x f(x, y)m(dy).$$

Proof. Let $h(x) := \int_E f(x, y)m(dy)$, $x \in \mathbb{R}^d$. By condition (i) and dominated convergence theorem, we have $h \in C_b^2(\mathbb{R}^d)$; in particular, $\nabla h(x) = \int_E \nabla_x f(x, y)m(dy)$. As a consequence, Ah is well-defined.

For $x \in \mathbb{R}^d$, we have

$$\begin{aligned} Ah(x) &= \gamma \cdot \nabla h(x) + \int_{\mathbb{R}^d \setminus \{0\}} (h(x+z) - h(x) - z \cdot \nabla h(x)) \nu(dz) \\ &= (\gamma - \int_{\{|z|>1\}} z \nu(dz)) \cdot \nabla h(x) + \int_{\{|z|>1\}} (h(x+z) - h(x)) \nu(dz) \\ &\quad + \int_{\{0<|z|\leq 1\}} (h(x+z) - h(x) - z \cdot \nabla h(x)) \nu(dz) \\ &= (\gamma - \int_{\{|z|>1\}} z \nu(dz)) \cdot \int_E \nabla_x f(x, y)m(dy) + I_1 + I_2, \end{aligned} \quad (3.7)$$

where

$$I_1 := \int_{\{|z|>1\}} (h(x+z) - h(x)) \nu(dz)$$

and

$$I_2 := \int_{\{0<|z|\leq 1\}} (h(x+z) - h(x) - z \cdot \nabla h(x)) \nu(dz).$$

Since $|f(x, \cdot)| \leq g_0$ for all $x \in \mathbb{R}^d$, we can apply Fubini's theorem to obtain

$$\begin{aligned} I_1 &= \int_{\{|z|>1\}} \int_E (f(x+z, y) - f(x, y))m(dy) \nu(dz) \\ &= \int_E \int_{\{|z|>1\}} (f(x+z, y) - f(x, y)) \nu(dz) m(dy). \end{aligned} \quad (3.8)$$

Noting that $\sum_{i,j=1}^d |\partial_{x_i x_j}^2 f(x, \cdot)| \leq g_2$ for all $x \in \mathbb{R}^d$, we can find a constant $C > 0$ such that

$$|\nabla_x f(x+z, y) - \nabla_x f(x, y)| \leq C g_2(y) |z|, \quad x, z \in \mathbb{R}^d, y \in E.$$

Therefore, for all $x, z \in \mathbb{R}^d$, $y \in E$,

$$\begin{aligned} |f(x+z, y) - f(x, y) - z \cdot \nabla_x f(x, y)| &\leq \int_0^1 |\nabla_x f(x+rz, y) - \nabla_x f(x, y)| |z| dr \\ &\leq C g_2(y) |z|^2. \end{aligned}$$

This allows us to use Fubini's theorem to obtain

$$\begin{aligned} I_2 &= \int_{\{0 < |z| \leq 1\}} \int_E (f(x+z, y) - f(x, y) - z \cdot \nabla_x f(x, y)) m(dy) \nu(dz) \\ &= \int_E \int_{\{0 < |z| \leq 1\}} (f(x+z, y) - f(x, y) - z \cdot \nabla_x f(x, y)) \nu(dz) m(dy). \end{aligned} \quad (3.9)$$

Now, the assertion follows easily from (3.7), (3.8) and (3.9). \square

Lemma 3.6. *Let $g \in C_b^2(\mathbb{R}^d)$ and $h_n \in C_0^\infty(\mathbb{R}^d)$ be such that $0 \leq h_n \leq 1$, $h_n(x) = 1$ for $|x| \leq n$ and $\sup_{n \in \mathbb{N}} \|h_n\|_{C_b^2(\mathbb{R}^d)} < \infty$. Then $A(gh_n) \rightarrow Ag$ boundedly and pointwise as $n \rightarrow \infty$.*

Proof. Firstly, we note that $\sup_{n \in \mathbb{N}} \|gh_n\|_{C_b^2(\mathbb{R}^d)} < \infty$. Therefore,

$$\begin{aligned} |(gh_n)(x+z) - (gh_n)(x) - z \cdot \nabla(gh_n)(x)| \\ \leq C(\mathbf{1}_{\{|z| > 1\}} + |z| \mathbf{1}_{\{|z| > 1\}} + |z|^2 \mathbf{1}_{\{|z| \leq 1\}}) \end{aligned} \quad (3.10)$$

for all $x, z \in \mathbb{R}^d$ and $n \in \mathbb{N}$, where $C > 0$ is a constant. Thus there exists $C' > 0$ such that $\|A(gh_n)\| \leq C'$. On the other hand, it is easy to verify that

$$\lim_{n \rightarrow \infty} (gh_n)(x) = g(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \nabla(gh_n)(x) = \nabla g(x) \quad (3.11)$$

for all $x \in \mathbb{R}^d$. Since the right-hand side of (3.10) is an integrable function with respect to the measure ν , by (3.11) and dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} A(gh_n)(x) = Ag(x), \quad x \in \mathbb{R}^d.$$

This completes the proof. \square

Remark 3.7. The existence of a sequence of functions $(h_n)_{n \in \mathbb{N}}$ being as in Lemma 3.6 is obvious. For example, we can take

$$g(t) := \begin{cases} \exp \frac{1}{(t-1)(t-4)}, & t \in (1, 4), \\ 0, & t \notin (1, 4). \end{cases}$$

Then let $F(t) := (\int_{-\infty}^{\infty} g(s) ds)^{-1} \int_t^{\infty} g(s) ds$ and $h_n(x) := F(|x|^2/n^2)$, $x \in \mathbb{R}^d$.

For $\lambda > 0$, the time-space resolvent operator R^λ of the stable process $S = (S_t)_{t \geq 0}$ is defined by

$$R^\lambda f(s, x) := \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y-x) f(t+s, y) dy dt, \quad (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad (3.12)$$

where $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ is an arbitrary measurable function such that the integral on the right of (3.12) is finite for all $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

The following proposition is about the solvability of the parabolic resolvent equation for the generator A of the stable process S . It plays a key role in obtaining a perturbative representation of the time-space resolvent of the solution to (1.1).

Proposition 3.8. *Suppose $\lambda > 0$ and $g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Let $f(s, x) := R^\lambda g(s, x)$, $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Then f belongs to $C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and solves the equation*

$$\lambda f - \partial_s f - A f = g \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \quad (3.13)$$

where A is defined by (2.3).

Proof. By the definition of R^λ , we have

$$f(s, x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y - x) g(t + s, y) dy dt \quad (3.14)$$

$$= \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) g(t + s, x + y) dy dt \quad (3.15)$$

for $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. In the rest of this proof we will use either the representation (3.14) or (3.15), according to our needs.

Since $g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, the functions $|g|, |\partial_t g|, |\nabla g|$ and $|\partial_{ij}^2 g|$, $i, j = 1, \dots, d$, are all bounded on $\mathbb{R}_+ \times \mathbb{R}^d$. It follows from (3.14) and dominated convergence theorem that $\partial_s f$ is bounded and continuous on $\mathbb{R}_+ \times \mathbb{R}^d$; moreover,

$$\partial_s f(s, x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y - x) \partial_s(g(t + s, y)) dy dt, \quad (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Similarly, by (3.15), ∇f and $\partial_{ij}^2 f$, $i, j = 1, \dots, d$, are also bounded and continuous. Thus $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Furthermore, it follows from (3.15) and Lemma 3.5 that

$$Af(s, x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) A_x(g(t + s, x + y)) dy dt. \quad (3.16)$$

We are now in a position to define an approximating sequence $(f_\epsilon)_{\epsilon>0}$ of f . In the following we will first derive an equation that f_ϵ fulfills, and then take the limit as $\epsilon \rightarrow 0$ to obtain (3.13) for f .

Let $\epsilon > 0$ and

$$f_\epsilon(s, x) := \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) g(t + s, x + y) dy dt.$$

Then

$$\partial_s f_\epsilon(s, x) = \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) \partial_s(g(t + s, x + y)) dy dt$$

and

$$\lim_{\epsilon \rightarrow 0} \partial_s f_\epsilon(s, x) = \partial_s f(s, x), \quad (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \quad (3.17)$$

Noting (3.3), it follows from Lemma 3.1 that $\lim_{t \rightarrow \infty} p_t(x) = 0$ for all $x \in \mathbb{R}^d$. By Fubini's theorem and integration by parts formula, we have

$$\begin{aligned}
\partial_s f_\epsilon(s, x) &= \int_{\mathbb{R}^d} \int_\epsilon^\infty e^{-\lambda t} p_t(y) \partial_s(g(t+s, x+y)) dt dy \\
&= \int_{\mathbb{R}^d} \int_\epsilon^\infty e^{-\lambda t} p_t(y) \partial_t(g(t+s, x+y)) dt dy \\
&= \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) g(t+s, x+y) \Big|_{t=\epsilon}^{t=\infty} dy \\
&\quad - \int_{\mathbb{R}^d} \int_\epsilon^\infty g(t+s, x+y) \partial_t(e^{-\lambda t} p_t(y)) dt dy \\
&= - \int_{\mathbb{R}^d} e^{-\lambda \epsilon} p_\epsilon(y) g(s+\epsilon, x+y) dy \\
&\quad + \int_{\mathbb{R}^d} \int_\epsilon^\infty e^{-\lambda t} g(t+s, x+y) (\lambda p_t(y) - \partial_t p_t(y)) dt dy \\
&=: I_\epsilon + J_\epsilon.
\end{aligned} \tag{3.18}$$

Obviously,

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} I_\epsilon &= - \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} p_\epsilon(y) g(s+\epsilon, x+y) dy \\
&= - \lim_{\epsilon \rightarrow 0} \mathbf{E}[g(s+\epsilon, x+S_\epsilon)] = -g(s, x).
\end{aligned} \tag{3.19}$$

By Fubini's theorem and a change of variables,

$$\begin{aligned}
J_\epsilon &= \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} g(t+s, x+y) (\lambda p_t(y) - \partial_t p_t(y)) dy dt \\
&= \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} (\lambda p_t(y-x) - \partial_t p_t(y-x)) g(t+s, y) dy dt.
\end{aligned} \tag{3.20}$$

Just as in (3.16), we have

$$A f_\epsilon(s, x) = \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) A_x(g(t+s, x+y)) dy dt,$$

so

$$\lim_{\epsilon \rightarrow 0} A f_\epsilon(s, x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) A_x(g(t+s, x+y)) dy dt = A f(s, x). \tag{3.21}$$

Let h_n be as in Lemma 3.6. By Lemma 3.5, Lemma 3.6 and dominated convergence theorem, we have

$$\begin{aligned}
Af_\epsilon(s, x) &= \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) A_x(g(t+s, x+y)) dy dt \\
&= \lim_{n \rightarrow \infty} \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) A_x(h_n(x+y)g(t+s, x+y)) dy dt \\
&= \lim_{n \rightarrow \infty} A_x \left(\int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y) h_n(x+y) g(t+s, x+y) dy dt \right) \\
&= \lim_{n \rightarrow \infty} A_x \left(\int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y-x) h_n(y) g(t+s, y) dy dt \right) \\
&= \lim_{n \rightarrow \infty} \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} A_x(p_t(y-x)) h_n(y) g(t+s, y) dy dt.
\end{aligned}$$

Since $|h_n| \leq 1$ and g is also bounded, it follows from Lemma 3.4 and dominated convergence theorem that

$$Af_\epsilon(s, x) = \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} A_x(p_t(y-x)) g(t+s, y) dy dt. \quad (3.22)$$

Finally, we verify that f —as the limit of f_ϵ —is a solution to the equation (3.13). By (3.18), (3.20) and (3.22), we have

$$\begin{aligned}
&(\lambda f_\epsilon - \partial_s f_\epsilon - Af_\epsilon)(s, x) \\
&= \lambda f_\epsilon(s, x) - I_\epsilon - J_\epsilon - \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} A_x(p_t(y-x)) g(t+s, y) dy dt \\
&= \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} (\partial_t p_t(y-x) - \lambda p_t(y-x) - A_x(p_t(y-x))) g(t+s, y) dy dt \\
&\quad + \lambda f_\epsilon(s, x) - I_\epsilon.
\end{aligned}$$

Since

$$\partial_t p_t(y-x) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(u) e^{-t\psi(u)} e^{-iu \cdot (y-x)} du, \quad x \in \mathbb{R}^d, t > 0,$$

it follows from (3.5) that $\partial_t p_t(y-x) = A_x(p_t(y-x))$, which implies

$$\begin{aligned}
&(\lambda f_\epsilon - \partial_s f_\epsilon - Af_\epsilon)(s, x) \\
&= \lambda f_\epsilon(s, x) - I_\epsilon - \lambda \int_\epsilon^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p_t(y-x) g(t+s, y) dy dt = -I_\epsilon. \quad (3.23)
\end{aligned}$$

Obviously, $f_\epsilon(s, x)$ converges to $f(s, x)$ as $\epsilon \rightarrow 0$. Letting $\epsilon \rightarrow 0$ in (3.23), the equation (3.13) follows from (3.17), (3.19) and (3.21). \square

Proposition 3.9. *Let $T > 0$ and $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function such that $\text{supp}(f) \subset [0, T] \times \mathbb{R}^d$.*

(i) If $f \in L^q([0, T]; L^p(\mathbb{R}^d))$ with $d/p + \alpha/q < \alpha$, then

$$\|R^\lambda f\| \leq N_\lambda \|f\|_{L^q([0, T]; L^p(\mathbb{R}^d))},$$

where $N_\lambda > 0$ is a constant depending on λ , p and q . Moreover, $N_\lambda \downarrow 0$ as $\lambda \rightarrow \infty$.
(ii) If $f \in L^q([0, T]; L^p(\mathbb{R}^d))$ with $d/p + \alpha/q < \alpha - 1$, then

$$\|\nabla R^\lambda f\| \leq M_\lambda \|f\|_{L^q([0, T]; L^p(\mathbb{R}^d))},$$

where $M_\lambda > 0$ is a constant depending on λ , p and q . Moreover, $M_\lambda \downarrow 0$ as $\lambda \rightarrow \infty$.

Proof. (i) Since $\text{supp}(f) \subset [0, T] \times \mathbb{R}^d$, upon using Hölder's inequality twice, we get

$$\begin{aligned} |R^\lambda f(s, x)| &= \left| \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^d} p_t(y - x) f(t + s, y) dy dt \right| \\ &\leq \int_0^\infty e^{-\lambda t} \|p_t(\cdot - x)\|_{L^{p^*}(\mathbb{R}^d)} \|f(t + s, \cdot)\|_{L^p(\mathbb{R}^d)} dt \\ &= \int_0^T e^{-\lambda t} \|p_t\|_{L^{p^*}(\mathbb{R}^d)} \|f(t + s, \cdot)\|_{L^p(\mathbb{R}^d)} dt \\ &\leq \left(\int_0^T e^{-q^* \lambda t} \|p_t\|_{L^{p^*}(\mathbb{R}^d)}^{q^*} dt \right)^{1/q^*} \|f\|_{L^q([0, T]; L^p(\mathbb{R}^d))}, \end{aligned}$$

where $p^*, q^* > 0$ are such that $1/p^* + 1/p = 1$ and $1/q^* + 1/q = 1$. By the scaling property (3.3),

$$\begin{aligned} \|p_t\|_{L^{p^*}(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} |p_t(y)|^{p^*} dy \right)^{1/p^*} \\ &= \left(\int_{\mathbb{R}^d} t^{-d(p^*-1)/\alpha} |p_1(y)|^{p^*} dy \right)^{1/p^*} \\ &= t^{-d(p^*-1)/(\alpha p^*)} \|p_1\|_{L^{p^*}(\mathbb{R}^d)}. \end{aligned}$$

Thus the assertion holds with

$$N_\lambda := \left(\int_0^\infty e^{-q^* \lambda t} t^{-dq^*(p^*-1)/(\alpha p^*)} dt \right)^{1/q^*} \|p_1\|_{L^{p^*}(\mathbb{R}^d)},$$

which is finite if $-dq^*(p^*-1)/(\alpha p^*) > -1$, or equivalently, $d/p + \alpha/q < \alpha$. By dominated convergence theorem, $\lim_{\lambda \rightarrow \infty} N_\lambda = 0$.

(ii) We first show that for fixed $t > 0$,

$$\nabla_x \left(\int_{\mathbb{R}^d} p_t(y - x) f(t + s, y) dy \right) = \int_{\mathbb{R}^d} \nabla_x(p_t(y - x)) f(t + s, y) dy. \quad (3.24)$$

To this end, set $f_n(t, y) := f(t, y) \mathbf{1}_{\{|y| \leq n\}}(y)$, $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d$. By dominated convergence theorem and Hölder's inequality,

$$\int_{\mathbb{R}^d} p_t(y - x) f_n(t + s, y) dy \rightarrow \int_{\mathbb{R}^d} p_t(y - x) f(t + s, y) dy \quad (3.25)$$

uniformly in $x \in \mathbb{R}^d$ as $n \rightarrow \infty$. Again by dominated convergence theorem,

$$\nabla_x \left(\int_{\mathbb{R}^d} p_t(y - x) f_n(t + s, y) dy \right) = \int_{\mathbb{R}^d} \nabla_x(p_t(y - x)) f_n(t + s, y) dy.$$

Just as in (3.25),

$$\int_{\mathbb{R}^d} \nabla_x(p_t(y-x))f_n(t+s, y)dy \rightarrow \int_{\mathbb{R}^d} \nabla_x(p_t(y-x))f(t+s, y)dy$$

uniformly in $x \in \mathbb{R}^d$ as $n \rightarrow \infty$. Since $\int_{\mathbb{R}^d} p_t(y-\cdot)f_n(t+s, y)dy \in C_b^1(\mathbb{R}^d)$ for each fixed $s, t \geq 0$ and $C_b^1(\mathbb{R}^d)$ is a Banach space, it follows that

$$\int_{\mathbb{R}^d} p_t(y-\cdot)f(t+s, y)dy \in C_b^1(\mathbb{R}^d)$$

and (3.24) holds.

For $i = 1, \dots, d$, by (3.3), we get

$$\begin{aligned} \|\partial_i p_t\|_{L^{p^*}(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} |\partial_{x_i} p_t(x)|^{p^*} dx \right)^{1/p^*} \\ &= \left(\int_{\mathbb{R}^d} t^{-p^*(d+1)/\alpha} |(\partial_i p_1)(t^{-1/\alpha}x + (1-t^{1-1/\alpha})\gamma)|^{p^*} dx \right)^{1/p^*} \\ &= \left(\int_{\mathbb{R}^d} t^{-p^*(d+1)/\alpha + d/\alpha} |\partial_{x_i} p_1(x)|^{p^*} dx \right)^{1/p^*} \\ &= t^{-(d+1)/\alpha + d/(\alpha p^*)} \|\partial_i p_1\|_{L^{p^*}(\mathbb{R}^d)}. \end{aligned}$$

As in (i), we can apply Hölder's inequality to obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \nabla_x(p_t(y-x))f(t+s, y)dy \right| &\leq \|\nabla_x(p_t(\cdot-x))\|_{L^{p^*}(\mathbb{R}^d)} \|f(t+s, \cdot)\|_{L^p(\mathbb{R}^d)} \\ &\leq \|\nabla p_t\|_{L^{p^*}(\mathbb{R}^d)} \|f(t+s, \cdot)\|_{L^p(\mathbb{R}^d)} \\ &\leq C t^{-(d+1)/\alpha + d/(\alpha p^*)} \|f(t+s, \cdot)\|_{L^p(\mathbb{R}^d)}, \quad (3.26) \end{aligned}$$

where $C > 0$ is a constant. If $d/p + \alpha/q < \alpha - 1$, then

$$-q^*(d+1)/\alpha + dq^*/(\alpha p^*) > -1.$$

Since $f \in L^q([0, T]; L^p(\mathbb{R}^d))$, by Hölder's inequality, we see that the right-hand side of (3.26) is an integrable function (with the variable t) on $[0, T]$. Now, it follows from (3.24), (3.26) and dominated convergence theorem that

$$\nabla R^\lambda f(s, x) = \int_0^\infty \exp(-\lambda t) \int_{\mathbb{R}^d} \nabla_x(p_t(y-x))f(t+s, y)dydt.$$

The rest of the proof is completely similar to that of (i), and we omit the details. We can take

$$M_\lambda := C \left(\int_0^\infty e^{-q^*\lambda t} t^{-q^*(d+1)/\alpha + dq^*/(\alpha p^*)} dt \right)^{1/q^*}.$$

□

Similarly to Proposition 3.9 (ii), we have the following estimate for R^λ . Its proof is very simple and is thus omitted.

Lemma 3.10. *For each $\lambda > 0$, there exists a constant $L_\lambda > 0$ such that*

$$\|\nabla R^\lambda g\| \leq L_\lambda \|g\|_{L^\infty([0, \infty); L^\infty(\mathbb{R}^d))}$$

for all $g \in L^\infty([0, \infty); L^\infty(\mathbb{R}^d))$, where $L_\lambda \downarrow 0$ as $\lambda \rightarrow \infty$.

4. Existence and Uniqueness of Weak Solutions: Local Case

In this section, we confine ourselves to the local case and thus assume in addition to Assumption 2.1 the following:

Assumption 4.1. The drift $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is such that $\text{supp}(b) \subset [0, T] \times \mathbb{R}^d$ and $b \in L^\infty([0, T]; L^\infty(\mathbb{R}^d)) + L^q([0, T]; L^p(\mathbb{R}^d))$ for some $T, p, q > 0$ with $d/p + \alpha/q < \alpha - 1$.

We first consider smooth approximations of the singular drift b . According to Assumption 4.1, we can assume $b = b_1 + b_2$ with $\text{supp}(b_i) \subset [0, T] \times \mathbb{R}^d$ for $i = 1, 2$, and

$$\|b_1\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^d))} \leq M, \quad \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))} < \infty, \quad (4.1)$$

where $M > 0$ is a constant and $d/p + \alpha/q < \alpha - 1$. Let

$$\hat{b}_{1,n} := b_1 \mathbf{1}_{\{|b_1| \leq n\}}, \quad \hat{b}_{2,n} := b_2 \mathbf{1}_{\{|b_2| \leq n\}}.$$

Next, for $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d$, define

$$b_1^{(n)}(t, y) := (\hat{b}_{1,n}(t, \cdot) * \varphi_n)(y), \quad b_2^{(n)}(t, y) := (\hat{b}_{2,n}(t, \cdot) * \varphi_n)(y), \quad (4.2)$$

where $(\varphi_n)_{n \in \mathbb{N}}$ is a mollifying sequence on \mathbb{R}^d . Then $b_1^{(n)}$ and $b_2^{(n)}$ are both bounded and globally Lipschitz in the space variable. Finally, let

$$b^{(n)} := b_1^{(n)} + b_2^{(n)}.$$

Obviously,

$$\text{supp}(b^{(n)}) \subset [0, T] \times \mathbb{R}^d \quad \text{and} \quad \|b^{(n)}\| \leq 2n. \quad (4.3)$$

Since $\|b_1\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^d))} \leq M$, it is easy to see that

$$\|b_1^{(n)}\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^d))} \leq M. \quad (4.4)$$

Remark 4.2. For each fixed $t \geq 0$, it follows from Young's inequality that

$$\|b_2^{(n)}(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq \|\hat{b}_{2,n}(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq \|b_2(t, \cdot)\|_{L^p(\mathbb{R}^d)}. \quad (4.5)$$

Therefore,

$$\|b_2^{(n)}\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}. \quad (4.6)$$

If $t \geq 0$ is such that $\|b_2(t, \cdot)\|_{L^p(\mathbb{R}^d)} < \infty$, then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|b_2^{(n)}(t, \cdot) - b_2(t, \cdot)\|_{L^p(\mathbb{R}^d)} \\ &= \lim_{n \rightarrow \infty} \|\hat{b}_{2,n}(t, \cdot) * \varphi_n - b_2(t, \cdot) * \varphi_n + b_2(t, \cdot) * \varphi_n - b_2(t, \cdot)\|_{L^p(\mathbb{R}^d)} \\ &\leq \limsup_{n \rightarrow \infty} \|\hat{b}_{2,n}(t, \cdot) - b_2(t, \cdot)\|_{L^p(\mathbb{R}^d)} + \limsup_{n \rightarrow \infty} \|b_2(t, \cdot) * \varphi_n - b_2(t, \cdot)\|_{L^p(\mathbb{R}^d)} \\ &= 0. \end{aligned} \quad (4.7)$$

It follows from (4.5), (4.7) and dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \|b_2^{(n)} - b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))} = 0. \quad (4.8)$$

Now, consider an α -stable ($1 < \alpha < 2$) process $S = (S_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$. As before, we assume that S fulfills Assumption 2.1, that is, S is non-degenerate. Recall that R^λ is the time-space resolvent of S and is defined in (3.12).

Define an operator BR^λ as follows. Given a function $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ for which $\nabla(R^\lambda f)$ is everywhere defined, define $BR^\lambda f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$BR^\lambda f(t, y) := b(t, y) \cdot \nabla R^\lambda f(t, y), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d. \quad (4.9)$$

For example, $BR^\lambda f$ is well-defined if $f \in L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$. Similarly, define $B_n R^\lambda f$ as

$$B_n R^\lambda f(t, y) := b^{(n)}(t, y) \cdot \nabla R^\lambda f(t, y), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad (4.10)$$

provided that $\nabla R^\lambda f$ exists everywhere.

Let M_λ and L_λ be as in Proposition 3.9 and Lemma 3.10, respectively. Since $M_\lambda \downarrow 0$ and $L_\lambda \downarrow 0$ as $\lambda \rightarrow \infty$, we can find $\lambda_0 > 0$ such that

$$L_{\lambda_0} M + M_{\lambda_0} \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))} < 1, \quad (4.11)$$

where $M > 0$ is the constant appearing in (4.1). If $\lambda > \lambda_0$, then $L_\lambda \leq L_{\lambda_0}$ and $M_\lambda \leq M_{\lambda_0}$. In view of (4.11), we have

$$\kappa_\lambda := L_\lambda M + M_\lambda \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))} < 1 \quad (4.12)$$

for any $\lambda > \lambda_0$.

Note that $b^{(n)}$ is bounded and globally Lipschitz in the space variable. We now consider an α -stable process with drift $b^{(n)}$.

Lemma 4.3. *Let $\lambda_0 > 0$ and κ_λ be as in (4.11) and (4.12), respectively. Suppose $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Let $X = (X_t)_{t \geq 0}$ be the unique strong solution to the SDE*

$$\begin{cases} dX_t = dS_t + b^{(n)}(s + t, X_t)dt, & t \geq 0, \\ X_0 = x. \end{cases} \quad (4.13)$$

Then for any $\lambda > \lambda_0$ and $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we have

$$\mathbf{E} \left[\int_0^\infty e^{-\lambda t} f(t + s, X_t) dt \right] = \sum_{k=0}^\infty R^\lambda (B_n R^\lambda)^k g(s, x), \quad (4.14)$$

where $B_n R^\lambda$ is defined by (4.10). Moreover, for each $k \in \mathbb{N}$,

$$\|R^\lambda (B_n R^\lambda)^k g\| \leq L_\lambda \|g\| (\kappa_\lambda)^{k-1} (M \lambda^{-1} + N_\lambda \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}), \quad (4.15)$$

which means that the series on the right-hand side of (4.14) converges and its convergence rate is independent of (s, x) and n .

Proof. For the existence and uniqueness of strong solutions to the SDE (4.13), the reader is referred to [8, Theorem 9.1] and [19, Theorem 117].

For $\lambda > 0$ and $f \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, define

$$V_n^\lambda f := \mathbf{E} \left[\int_0^\infty e^{-\lambda t} f(t + s, X_t) dt \right].$$

Applying Itô's formula for $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, we obtain

$$\begin{aligned} & f(t+s, X_t) - f(s, X_0) \\ &= \text{"Martingale"} + \int_0^t \left(\frac{\partial f}{\partial u} + L_u^{(n)} f \right)(u+s, X_u) du, \end{aligned}$$

where $L_u^{(n)} := A + b^{(n)}(u, \cdot) \cdot \nabla$ for $u \geq 0$. Taking expectations of both sides of the above equality gives

$$\mathbf{E}[f(t+s, X_t)] - f(s, x) = \mathbf{E} \left[\int_0^t \left(\frac{\partial f}{\partial u} + L_u^{(n)} f \right)(u+s, X_u) du \right]. \quad (4.16)$$

Note that

$$\mathbf{E} \left[\int_0^\infty e^{-\lambda t} |b^{(n)}(t+s, X_t)| dt \right] < \infty.$$

Multiplying both sides of (4.16) by $e^{-\lambda t}$, integrating with respect to t from 0 to ∞ and then applying Fubini's theorem, we get

$$\begin{aligned} & \mathbf{E} \left[\int_0^\infty e^{-\lambda t} f(t+s, X_t) dt \right] \\ &= \frac{1}{\lambda} f(s, x) + \mathbf{E} \left[\int_0^\infty e^{-\lambda t} \int_0^t \left(\frac{\partial f}{\partial u} + L_u^{(n)} f \right)(u+s, X_u) du dt \right] \\ &= \frac{1}{\lambda} f(s, x) + \frac{1}{\lambda} \mathbf{E} \left[\int_0^\infty e^{-\lambda u} \left(\frac{\partial f}{\partial u} + L_u^{(n)} f \right)(u+s, X_u) du \right]. \end{aligned}$$

Therefore, for $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\lambda V_n^\lambda f = f(s, x) + V_n^\lambda \left(\frac{\partial f}{\partial t} + L_t^{(n)} f \right). \quad (4.17)$$

Given $g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, it follows from Proposition 3.8 that $f := R^\lambda g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and $(\lambda - A - \frac{\partial}{\partial t})f = g$. Substituting this f in (4.17), we obtain

$$V_n^\lambda g = R^\lambda g(s, x) + V_n^\lambda (B_n R^\lambda g) \quad (4.18)$$

for $g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. After a standard approximation procedure, the equality (4.18) holds for any bounded continuous function g on $\mathbb{R}_+ \times \mathbb{R}^d$. For any open subset O of $\mathbb{R}_+ \times \mathbb{R}^d$, we can find $f_k \in C_b(\mathbb{R}_+ \times \mathbb{R}^d)$, $k \in \mathbb{N}$, such that $0 \leq f_k \uparrow \mathbf{1}_O$ as $k \rightarrow \infty$. It is easy to see that $R^\lambda f_k$ and $\nabla R^\lambda f_k$ converge boundedly and pointwise to $R^\lambda \mathbf{1}_O$ and $\nabla R^\lambda \mathbf{1}_O$, respectively. By dominated convergence theorem, (4.18) holds for $g = \mathbf{1}_O$. Then we can use a monotone class argument (see, for example, [4, p. 4]) to extend (4.18) to every $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$.

Therefore, we have shown that

$$V_n^\lambda f - R^\lambda f(s, x) = V_n^\lambda (B_n R^\lambda f), \quad f \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d). \quad (4.19)$$

For any bounded measurable function g on $\mathbb{R}_+ \times \mathbb{R}^d$, taking $f = B_n R^\lambda g$ in (4.19), we get

$$V_n^\lambda (B_n R^\lambda g) - R^\lambda B_n R^\lambda g(s, x) = V_n^\lambda (B_n R^\lambda)^2 g$$

and thus

$$\begin{aligned} V_n^\lambda g &= R^\lambda g(s, x) + V_n^\lambda(B_n R^\lambda g) \\ &= R^\lambda g(s, x) + R^\lambda B_n R^\lambda g(s, x) + V_n^\lambda(B_n R^\lambda)^2 g. \end{aligned}$$

Similarly, after i steps, we obtain

$$V_n^\lambda g = \sum_{k=0}^i R^\lambda(B_n R^\lambda)^k g(s, x) + V_n^\lambda(B_n R^\lambda)^{i+1} g, \quad g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d). \quad (4.20)$$

In order to show that the first term on the right-hand side of (4.20) converges as $i \rightarrow \infty$, we first need to prove the following claim.

Claim 1. Suppose that $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that $\|\nabla R^\lambda g\| < \infty$. Then for each $k \in \mathbb{Z}_+$,

$$\|\nabla R^\lambda(B_n R^\lambda)^k g\| \leq \|\nabla R^\lambda g\|(\kappa_\lambda)^k \quad (4.21)$$

and

$$|(B_n R^\lambda)^{k+1} g| \leq \|\nabla R^\lambda g\|(\kappa_\lambda)^k (|b_1^{(n)}| + |b_2^{(n)}|). \quad (4.22)$$

We prove Claim 1 by induction. If $k = 0$, then (4.21) is trivial and

$$|B_n R^\lambda g| \leq \|\nabla R^\lambda g\| |b^{(n)}| \leq \|\nabla R^\lambda g\|(|b_1^{(n)}| + |b_2^{(n)}|).$$

Suppose now that the above claim is true for k . Note that (4.3), (4.4) and (4.6) hold. By Proposition 3.9 and Lemma 3.10, we get

$$\begin{aligned} \|\nabla R^\lambda(B_n R^\lambda)^{k+1} g\| &= \|\nabla R^\lambda(b^{(n)} \cdot \nabla R^\lambda(B_n R^\lambda)^k g)\| \\ &\leq \|\nabla R^\lambda(b_1^{(n)} \cdot \nabla R^\lambda(B_n R^\lambda)^k g)\| + \|\nabla R^\lambda(b_2^{(n)} \cdot \nabla R^\lambda(B_n R^\lambda)^k g)\| \\ &\leq L_\lambda \|b_1^{(n)} \cdot \nabla R^\lambda(B_n R^\lambda)^k g\|_{L^\infty([0, \infty); L^\infty(\mathbb{R}^d))} \\ &\quad + M_\lambda \|b_2^{(n)} \cdot \nabla R^\lambda(B_n R^\lambda)^k g\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \\ &\leq \|\nabla R^\lambda g\|(\kappa_\lambda)^k (L_\lambda M + M_\lambda \|b_2^{(n)}\|_{L^q([0, T]; L^p(\mathbb{R}^d))}) \\ &\leq \|\nabla R^\lambda g\|(\kappa_\lambda)^k (L_\lambda M + M_\lambda \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}) \\ &= \|\nabla R^\lambda g\|(\kappa_\lambda)^{k+1} \end{aligned}$$

and

$$|(B_n R^\lambda)^{k+2} g| \leq \|\nabla R^\lambda g\|(\kappa_\lambda)^{k+1} (|b_1^{(n)}| + |b_2^{(n)}|).$$

Thus the claim is also true for $k + 1$. Hence Claim 1 is true for any $k \in \mathbb{Z}_+$.

Note that $\|R^\lambda f\| \leq \lambda^{-1} \|f\|_{L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d))}$ for all $f \in L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$. By (4.22) and Proposition 3.9, we obtain

$$\begin{aligned} \|R^\lambda(B_n R^\lambda)^k g\| &\leq \|\nabla R^\lambda g\|(\kappa_\lambda)^{k-1} R^\lambda(|b_1^{(n)}| + |b_2^{(n)}|) \\ &\leq \|\nabla R^\lambda g\|(\kappa_\lambda)^{k-1} (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}). \end{aligned} \quad (4.23)$$

If $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, then $\|\nabla R^\lambda g\| \leq L_\lambda \|g\|$ by Lemma 3.10. So the inequality (4.15) is proved. By (4.12) and (4.15), we see that the series $\sum_{k=0}^\infty R^\lambda(B_n R^\lambda)^k g$ converges uniformly on $\mathbb{R}_+ \times \mathbb{R}^d$ for any $\lambda > \lambda_0$ and $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$.

Finally, we show that the second term on the right-hand side of (4.20) converges to 0 as $i \rightarrow \infty$. Note that $|b_1^{(n)}|$ and $|b_2^{(n)}|$ are both bounded by n . According to Claim 1, we have, for any $\lambda > \lambda_0$ and $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\begin{aligned} |V_n^\lambda(B_n R^\lambda)^{i+1} g| &\leq \|\nabla R^\lambda g\|(\kappa_\lambda)^i V_n^\lambda(|b_1^{(n)}| + |b_2^{(n)}|) \\ &\leq 2n\lambda^{-1} L_\lambda \|g\|(\kappa_\lambda)^i, \end{aligned}$$

which converges to 0 as $i \rightarrow \infty$. Now, the equality (4.14) follows from (4.15) and (4.20). This completes the proof. \square

In view of (4.15), we can define an operator G_n^λ on $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ as

$$G_n^\lambda g = \sum_{k=0}^{\infty} R^\lambda(B_n R^\lambda)^k g, \quad g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d), \quad (4.24)$$

provided that $\lambda > \lambda_0$. In the next lemma we study the limiting behavior of G_n^λ as $n \rightarrow \infty$.

Lemma 4.4. *Let $\lambda > \lambda_0$. For $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, define*

$$G^\lambda g := \sum_{k=0}^{\infty} R^\lambda(BR^\lambda)^k g. \quad (4.25)$$

(i) *Then the series on the right-hand side of (4.25) converges uniformly on $\mathbb{R}_+ \times \mathbb{R}^d$ for any $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$.*

(ii) *For each $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, $G_n^\lambda g$ converges locally uniformly to $G^\lambda g$ as $n \rightarrow \infty$, that is, for any compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$,*

$$\lim_{n \rightarrow \infty} \sup_{(s,x) \in K} |G_n^\lambda g(s,x) - G^\lambda g(s,x)| = 0.$$

Proof. (i) Let $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$. With the same argument that we used to establish (4.21) and (4.23), we conclude that, for each $k \in \mathbb{N}$,

$$\|\nabla R^\lambda(BR^\lambda)^k g\| \leq \|\nabla R^\lambda g\|(\kappa_\lambda)^k \quad (4.26)$$

and

$$\|R^\lambda(BR^\lambda)^k g\| \leq \|\nabla R^\lambda g\|(\kappa_\lambda)^{k-1} (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}). \quad (4.27)$$

As before, we have $\|\nabla R^\lambda g\| \leq L_\lambda \|g\|$ by Lemma 3.10. Noting (4.12), we see that the series $\sum_{k=0}^{\infty} R^\lambda(BR^\lambda)^k g$ converges uniformly on $\mathbb{R}_+ \times \mathbb{R}^d$ for any $\lambda > \lambda_0$.

(ii) Suppose $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$. By (4.12) and the estimates (4.15) and (4.27), we only need to show the following claim.

Claim 2. For any fixed $k \in \mathbb{Z}_+$ and compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$,

$$\lim_{n \rightarrow \infty} \sup_{(s,x) \in K} |R^\lambda(B_n R^\lambda)^k g(s,x) - R^\lambda(BR^\lambda)^k g(s,x)| = 0 \quad (4.28)$$

and

$$\lim_{n \rightarrow \infty} \sup_{(s,x) \in K} |\nabla R^\lambda(B_n R^\lambda)^k g(s,x) - \nabla R^\lambda(BR^\lambda)^k g(s,x)| = 0. \quad (4.29)$$

We prove Claim 2 by induction. If $k = 0$, then (4.28) and (4.29) are trivially true. Suppose that the above claim is true for k . For $m > 0$ let

$$A_m := \{x \in \mathbb{R}^d : |x| \leq m\}. \quad (4.30)$$

Define $h_m : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$h_m(t, y) := \mathbf{1}_{A_m}(y), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d. \quad (4.31)$$

Now set

$$C_{n,m} := \sup_{(s,x) \in [0,T] \times A_m} |\nabla R^\lambda(B_n R^\lambda)^k g(s, x) - \nabla R^\lambda(BR^\lambda)^k g(s, x)|.$$

By induction hypothesis, $\lim_{n \rightarrow \infty} C_{n,m} = 0$ for any $m > 0$.

Since the support of $b^{(n)}$ and b are both contained in $[0, T] \times \mathbb{R}^d$, it follows that

$$\nabla R^\lambda(B_n R^\lambda)^k g(s, x) = \nabla R^\lambda(BR^\lambda)^k g(s, x) = 0, \quad \forall s > T, x \in \mathbb{R}^d.$$

By (4.21) and (4.26), we have

$$\begin{aligned} & |(B_n R^\lambda)^{k+1} g - (BR^\lambda)^{k+1} g| \\ & \leq |(B_n R^\lambda)^{k+1} g - B_n R^\lambda (BR^\lambda)^k g| + |B_n R^\lambda (BR^\lambda)^k g - (BR^\lambda)^{k+1} g| \\ & \leq |\nabla R^\lambda(B_n R^\lambda)^k g - \nabla R^\lambda(BR^\lambda)^k g| |b^{(n)}| + \|\nabla R^\lambda(BR^\lambda)^k g\| |b^{(n)} - b| \\ & = |\nabla R^\lambda(B_n R^\lambda)^k g - \nabla R^\lambda(BR^\lambda)^k g| |b^{(n)}| (h_m + (1 - h_m)) \\ & \quad + \|\nabla R^\lambda(BR^\lambda)^k g\| |b^{(n)} - b| \\ & \leq C_{n,m} |b^{(n)}| + 2C |b^{(n)}| (1 - h_m) + C |b^{(n)} - b|, \end{aligned}$$

where

$$C := \|\nabla R^\lambda g\| (\kappa_\lambda)^k \leq L_\lambda \|g\| (\kappa_\lambda)^k < \infty$$

is a constant. Therefore,

$$\begin{aligned} & |R^\lambda(B_n R^\lambda)^{k+1} g - R^\lambda(BR^\lambda)^{k+1} g| \\ & \leq C_{n,m} R^\lambda(|b^{(n)}|) + 2C R^\lambda(|b^{(n)}| (1 - h_m)) + C R^\lambda(|b^{(n)} - b|) \\ & \leq C_{n,m} \|R^\lambda(|b^{(n)}|)\| + 2C R^\lambda(|b^{(n)}| (1 - h_m)) + C R^\lambda(|b^{(n)} - b|). \end{aligned} \quad (4.32)$$

By (4.4), (4.6) and Proposition 3.9, we have

$$\begin{aligned} \sup_{n \in \mathbb{N}} \|R^\lambda(|b^{(n)}|)\| & \leq \sup_{n \in \mathbb{N}} \|R^\lambda(|b_1^{(n)}| + |b_2^{(n)}|)\| \\ & \leq \sup_{n \in \mathbb{N}} (\lambda^{-1} M + N_\lambda \|b_2^{(n)}\|_{L^q([0,T]; L^p(\mathbb{R}^d))}) \\ & \leq \lambda^{-1} M + N_\lambda \|b_2\|_{L^q([0,T]; L^p(\mathbb{R}^d))} < \infty, \end{aligned} \quad (4.33)$$

which implies

$$\lim_{n \rightarrow \infty} C_{n,m} \|R^\lambda(|b^{(n)}|)\| = 0, \quad \forall m > 0. \quad (4.34)$$

Similarly,

$$\begin{aligned} R^\lambda(|b^{(n)} - b|) & \leq R^\lambda(|b_1^{(n)} - b_1| + |b_2^{(n)} - b_2|) \\ & \leq R^\lambda(|b_1^{(n)} - b_1|) + N_\lambda \|b_2^{(n)} - b_2\|_{L^q([0,T]; L^p(\mathbb{R}^d))}. \end{aligned} \quad (4.35)$$

For $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\begin{aligned}
 & R^\lambda(|b_1^{(n)} - b_1|)(s, x) \\
 &= R^\lambda(|b_1^{(n)} - b_1|h_m)(s, x) + R^\lambda(|b_1^{(n)} - b_1|(1 - h_m))(s, x) \\
 &\leq N_\lambda \|(|b_1^{(n)} - b_1|)h_m\|_{L^q([0, T]; L^p(\mathbb{R}^d))} + 2M \int_0^\infty \int_{\{|y|>m\}} e^{-\lambda t} p_t(y - x) dy dt \\
 &=: I_{n,m} + J_m(x).
 \end{aligned} \tag{4.36}$$

Similarly to (4.8), for any fixed $m > 0$,

$$\lim_{n \rightarrow \infty} I_{n,m} = 0, \quad \forall m > 0. \tag{4.37}$$

If (s, x) is in the compact set K and $m > 0$ is sufficiently large, then

$$\begin{aligned}
 J_m(x) &= 2M \int_0^\infty \int_{\{|y'+x|>m\}} e^{-\lambda t} p_t(y') dy' dt \\
 &\leq 2M \int_0^\infty \int_{\{|y'|>m/2\}} e^{-\lambda t} p_t(y') dy' dt.
 \end{aligned}$$

By dominated convergence theorem,

$$\lim_{m \rightarrow \infty} \sup_{(s,x) \in K} J_m(x) = 0. \tag{4.38}$$

In view of (4.36), (4.37) and (4.38), we can use a simple “ $\epsilon - \delta$ ” argument to obtain

$$\lim_{n \rightarrow \infty} \sup_{(s,x) \in K} R^\lambda(|b_1^{(n)} - b_1|)(s, x) = 0. \tag{4.39}$$

It follows from (4.8), (4.35) and (4.39) that

$$\lim_{n \rightarrow \infty} \sup_{(s,x) \in K} R^\lambda(|b^{(n)} - b|)(s, x) = 0. \tag{4.40}$$

We now turn to treat the second term on the right-hand side of (4.32). For $(s, x) \in K$ and sufficiently large $m > 0$, we have

$$\begin{aligned}
 & R^\lambda(|b^{(n)}|(1 - h_m))(s, x) \\
 &\leq R^\lambda(|b_1^{(n)}|(1 - h_m))(s, x) + R^\lambda(|b_2^{(n)}|(1 - h_m))(s, x) \\
 &\leq M \int_0^\infty \int_{\{|y|>m\}} e^{-\lambda t} p_t(y - x) dy dt \\
 &\quad + \int_0^\infty e^{-\lambda t} \|p_t(\cdot - x)(1 - h_m)\|_{L^{p^*}(\mathbb{R}^d)} \|b_2^{(n)}(s + t, \cdot)\|_{L^p(\mathbb{R}^d)} dt \\
 &\leq \frac{1}{2} J_m(x) + \int_0^\infty e^{-\lambda t} \|(1 - h_{m/2})p_t\|_{L^{p^*}(\mathbb{R}^d)} \|b_2(s + t, \cdot)\|_{L^p(\mathbb{R}^d)} dt \\
 &\leq \frac{1}{2} J_m(x) + \int_0^T \|(1 - h_{m/2})p_t\|_{L^{p^*}(\mathbb{R}^d)} \|b_2(s + t, \cdot)\|_{L^p(\mathbb{R}^d)} dt \\
 &\leq \frac{1}{2} J_m(x) + \|(1 - h_{m/2})p_t\|_{L^{q^*}([0, T]; L^{p^*}(\mathbb{R}^d))} \|b_2\|_{L^q([0, T]; L^p(\mathbb{R}^d))}.
 \end{aligned}$$

By (4.38) and dominated convergence theorem, we see that

$$\lim_{m \rightarrow \infty} \sup_{\substack{(s,x) \in K \\ n \in \mathbb{N}}} R^\lambda(|b^{(n)}|(1 - h_m))(s, x) = 0. \quad (4.41)$$

For any $\epsilon > 0$, we can find large enough $m_0 > 0$ such that

$$\sup_{\substack{(s,x) \in K \\ n \in \mathbb{N}}} R^\lambda(|b^{(n)}|(1 - h_{m_0}))(s, x) < \frac{\epsilon}{6C}.$$

By (4.34) and (4.40), there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$C_{n,m_0} \|R^\lambda(|b^{(n)}|)\| < \frac{\epsilon}{3} \quad \text{and} \quad \sup_{(s,x) \in K} R^\lambda(|b^{(n)} - b|)(s, x) < \frac{\epsilon}{3C}.$$

It follows from (4.32) that

$$\sup_{(s,x) \in K} |R^\lambda(B_n R^\lambda)^{k+1} g(s, x) - R^\lambda(B R^\lambda)^{k+1} g(s, x)| < \epsilon, \quad \forall n \geq n_0,$$

which shows (4.28) for $k + 1$.

Similarly, we can show that (4.29) is also true for $k + 1$. Hence Claim 2 is true for any $k \in \mathbb{Z}_+$. This completes the proof. \square

Remark 4.5. By (4.27), Proposition 3.9 and Lemma 3.10, we have

$$\begin{aligned} \|R^\lambda(B R^\lambda)^k(|b|)\| &\leq \|\nabla R^\lambda(|b|)\| (\kappa_\lambda)^{k-1} (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}) \\ &= \|\nabla R^\lambda(|b_1 + b_2|)\| (\kappa_\lambda)^{k-1} (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}) \\ &\leq (\kappa_\lambda)^k (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}). \end{aligned}$$

For $\lambda > \lambda_0$, $G^\lambda(|b|) := \sum_{k=0}^{\infty} R^\lambda(B R^\lambda)^k(|b|)$ is thus well-defined and

$$\|G^\lambda(|b|)\| \leq \sum_{k=0}^{\infty} \|R^\lambda(B R^\lambda)^k(|b|)\| < \infty.$$

Lemma 4.6. *As $\lambda \rightarrow \infty$, $G_n^\lambda(|b^{(n)}|)(s, x)$ converges to 0 uniformly in $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $n \in \mathbb{N}$.*

Proof. Let $m, n \in \mathbb{N}$ and $\lambda > \lambda_0$. Since $|b^{(m)}|$ is bounded, by (4.24),

$$G_n^\lambda(|b^{(m)}|) = \sum_{k=0}^{\infty} R^\lambda(B_n R^\lambda)^k(|b^{(m)}|).$$

Since $|b^{(m)}| \leq |b_1^{(m)}| + |b_2^{(m)}|$, by (4.22), Proposition 3.9 and Lemma 3.10, for each $k \in \mathbb{Z}_+$,

$$\begin{aligned} |(B_n R^\lambda)^{k+1}(|b^{(m)}|)| &\leq \|\nabla R^\lambda(|b^{(m)}|)\| (\kappa_\lambda)^k (|b_1^{(n)}| + |b_2^{(n)}|) \\ &\leq (\kappa_\lambda)^{k+1} (|b_1^{(n)}| + |b_2^{(n)}|). \end{aligned}$$

Therefore,

$$\begin{aligned}
G_n^\lambda(|b^{(m)}|) &\leq R^\lambda(|b^{(m)}|) + \sum_{k=1}^\infty (\kappa_\lambda)^k R^\lambda(|b_1^{(n)}| + |b_2^{(n)}|) \\
&\leq R^\lambda(|b_1^{(m)}| + |b_2^{(m)}|) + \sum_{k=1}^\infty (\kappa_\lambda)^k (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}) \\
&\leq (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}) \sum_{k=0}^\infty (\kappa_\lambda)^k \\
&= (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}) (1 - \kappa_\lambda)^{-1}.
\end{aligned} \tag{4.42}$$

Since $N_\lambda \downarrow 0$ as $\lambda \rightarrow \infty$, it follows that $G_n^\lambda(|b^{(n)}|)(s, x)$ converges to 0 uniformly in $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $n \in \mathbb{N}$ as $\lambda \rightarrow \infty$. \square

We are now ready to prove the local weak existence for the SDE (1.1).

Theorem 4.7. *Let $d \geq 2$ and $1 < \alpha < 2$. Assume Assumption 4.1. Then for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there exists a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, on which a non-degenerate α -stable process $(S_t)_{t \geq 0}$ and a càdlàg process $(X_t)_{t \geq 0}$ are defined, such that (1.1) is satisfied and*

$$\mathbf{E} \left[\int_0^\infty e^{-\lambda t} g(t + s, X_t) dt \right] = G^\lambda g(s, x), \quad \lambda > \lambda_0, \quad g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d), \tag{4.43}$$

where G^λ is defined in (4.25).

Proof. For the construction of weak solutions to (1.1), we basically follow the proofs of [25, Theorem 4.1] and [14, Theorem 3.1]. We will see that the weak solutions constructed in this way would automatically satisfy (4.43).

Since $b^{(n)}(\cdot, \cdot)$ is bounded and globally Lipschitz in the space variable, for any non-degenerate α -stable process S defined on a filtered probability space $(\Omega, \mathcal{A}, \mathbf{P})$, there exists a strong solution $(X_t^n)_{t \geq 0}$ to the SDE

$$\begin{cases} dX_t^n = dS_t + b^{(n)}(s + t, X_t^n) dt, & t \geq 0, \\ X_0 = x. \end{cases}$$

Therefore, for each $t \geq 0$,

$$X_t^n = x + S_t + \int_0^t b^{(n)}(s + u, X_u^n) du \quad \text{a.s.} \tag{4.44}$$

Define $Y := (Y_t^n)_{t \geq 0}$ with $Y_t^n := \int_0^t b^{(n)}(s + u, X_u^n) du$, $t \geq 0$, and $Z^n := (X^n, Y^n, S)$. Since the remaining proof is rather long, we do it into several steps.

“Step 1”: We show that the family $\{Z^n : n \in \mathbb{N}\}$ of random elements in (D^3, \mathcal{D}^3) is tight. It suffices to show that

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq u \leq t} |Z_u^n| > l \right) = 0, \quad \forall t \geq 0, \tag{4.45}$$

and

$$\limsup_{n \rightarrow \infty} \mathbf{P}(|Z_{t \wedge (\tau^n + r_n)}^n - Z_{t \wedge \tau^n}^n| > \epsilon) = 0, \quad \forall t \geq 0, \epsilon > 0, \tag{4.46}$$

where each τ^n is an stopping-time with respect to the natural filtration induced by Z^n , $n \in \mathbb{N}$, and $(r_n)_{n \in \mathbb{N}}$ is a sequence of real numbers with $r_n \downarrow 0$ as $n \rightarrow \infty$.

For $0 \leq u \leq t$, we have

$$|Z_u^n| \leq \sqrt{3} \left(|x| + |S_u| + \int_0^u |b^{(n)}(s+r, X_r^n)| dr \right),$$

so

$$\sup_{0 \leq u \leq t} |Z_u^n| \leq \sqrt{3} \left(|x| + \sup_{0 \leq u \leq t} |S_u| + \int_0^t |b^{(n)}(s+r, X_r^n)| dr \right).$$

Therefore, to get (4.45), it suffices to show

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\int_0^t |b^{(n)}(s+u, X_u^n)| du > l \right) = 0, \quad \forall t \geq 0. \quad (4.47)$$

By Chebyshev's inequality and Lemma 4.3, we have

$$\begin{aligned} \mathbf{P} \left(\int_0^t |b^{(n)}(s+u, X_u^n)| du > l \right) &\leq l^{-1} \mathbf{E} \left[\int_0^t |b^{(n)}(s+u, X_u^n)| du \right] \\ &\leq l^{-1} e^{\theta t} \mathbf{E} \left[\int_0^t e^{-\theta u} |b^{(n)}(s+u, X_u^n)| du \right] \\ &\leq l^{-1} e^{\theta t} \mathbf{E} \left[\int_0^\infty e^{-\theta u} |b^{(n)}(s+u, X_u^n)| du \right] \\ &\leq l^{-1} e^{\theta t} G_n^\theta(|b^{(n)}|)(s, x), \end{aligned} \quad (4.48)$$

where $\theta > \lambda_0$ is a constant. By (4.42), $\sup_{n \in \mathbb{N}} G_n^\theta(|b^{(n)}|)(s, x) < \infty$. So (4.47) follows from (4.48). As a consequence, (4.45) is proved.

By (4.44), we have

$$|Z_{t \wedge (\tau^n + r_n)}^n - Z_{t \wedge \tau^n}^n| \leq 2\sqrt{3} \left(|S_{t \wedge (\tau^n + r_n)} - S_{t \wedge \tau^n}| + \int_{t \wedge \tau^n}^{t \wedge (\tau^n + r_n)} |b^{(n)}(s+u, X_u^n)| du \right),$$

so

$$\begin{aligned} \mathbf{P}(|Z_{t \wedge (\tau^n + r_n)}^n - Z_{t \wedge \tau^n}^n| > \epsilon) &\leq \mathbf{P} \left(|S_{t \wedge (\tau^n + r_n)} - S_{t \wedge \tau^n}| > \frac{\epsilon}{4\sqrt{3}} \right) \\ &\quad + \mathbf{P} \left(\int_{t \wedge \tau^n}^{t \wedge (\tau^n + r_n)} |b^{(n)}(s+u, X_u^n)| du > \frac{\epsilon}{4\sqrt{3}} \right). \end{aligned} \quad (4.49)$$

Since $S_0 = 0$ a.s. and S has càdlàg paths, it follows from strong Markov property that

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left(|S_{t \wedge (\tau^n + r_n)} - S_{t \wedge \tau^n}| > \frac{\epsilon}{4\sqrt{3}} \right) \leq \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq u \leq r_n} |S_u| > \frac{\epsilon}{4\sqrt{3}} \right) = 0. \quad (4.50)$$

Again by the strong Markov property,

$$\begin{aligned} &\mathbf{P} \left(\int_{t \wedge \tau^n}^{t \wedge (\tau^n + r_n)} |b^{(n)}(s+u, X_u^n)| du > \frac{\epsilon}{4\sqrt{3}} \right) \\ &= \mathbf{E} \left[\mathbf{P} \left(\int_{t \wedge \tau^n}^{t \wedge (\tau^n + r_n)} |b^{(n)}(s+u, X_u^n)| du > \frac{\epsilon}{4\sqrt{3}} \mid X_{t \wedge \tau^n}^n \right) \right]. \end{aligned} \quad (4.51)$$

By Chebyshev's inequality and Lemma 4.3,

$$\begin{aligned}
& \mathbf{P}\left(\int_{t \wedge \tau^n}^{t \wedge (\tau^n + r_n)} |b^{(n)}(s+u, X_u^n)| du > \frac{\epsilon}{4\sqrt{3}} \mid X_{t \wedge \tau^n}^n\right) \\
& \leq 4\sqrt{3}\epsilon^{-1} \mathbf{E}\left[\int_{t \wedge \tau^n}^{t \wedge \tau^n + r_n} |b^{(n)}(s+u, X_u^n)| du \mid X_{t \wedge \tau^n}^n\right] \\
& \leq 4\sqrt{3}\epsilon^{-1} \mathbf{E}\left[\int_{t \wedge \tau^n}^{t \wedge \tau^n + r_n} \exp\left(-\frac{u - t \wedge \tau^n}{r_n}\right) |b^{(n)}(s+u, X_u^n)| du \mid X_{t \wedge \tau^n}^n\right] \\
& \leq 4\sqrt{3}\epsilon^{-1} G_n^{\lambda_n}(|b^{(n)}|)(X_{t \wedge \tau^n}^n) \\
& \leq 4\sqrt{3}\epsilon^{-1} \|G_n^{\lambda_n}(|b^{(n)}|)\|,
\end{aligned} \tag{4.52}$$

where $\lambda_n := 1/r_n$. By Lemma 4.6, we have $\lim_{n \rightarrow \infty} \|G_n^{\lambda_n}(|b^{(n)}|)\| = 0$. It follows from (4.51) and (4.52) that

$$\limsup_{n \rightarrow \infty} \mathbf{P}\left(\int_{t \wedge \tau^n}^{t \wedge (\tau^n + r_n)} |b^{(n)}(s+u, X_u^n)| du > \frac{\epsilon}{4\sqrt{3}}\right) = 0. \tag{4.53}$$

Combining (4.49), (4.50) and (4.53), we obtain (4.46).

As shown in the proof of [14, Theorem 3.1], we can use the conditions (4.45) and (4.46) to find a probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbf{P}})$ and processes $\tilde{Z} = (\tilde{X}, \tilde{Y}, \tilde{S})$, $\tilde{Z}_n = (\tilde{X}^n, \tilde{Y}^n, \tilde{S}^n)$, $n = 1, 2, \dots$, defined on it, such that

- (i) $\tilde{Z}_n \rightarrow \tilde{Z}$ $\tilde{\mathbf{P}}$ -a.s. (as random elements in (D^3, \mathcal{D}^3)) as $n \rightarrow \infty$.
- (ii) \tilde{Z}_n and Z_n are identically distributed for each $n \in \mathbb{N}$.
- (iii) $\tilde{Z}, \tilde{Z}_n, n = 1, 2, \dots$, have càdlàg paths.

In fact, the above three properties hold only for a subsequence $\tilde{Z}_{n_k}, k \in \mathbb{N}$; however, for simplicity, we denote this subsequence still by $\tilde{Z}_n, n \in \mathbb{N}$. It is easy to see that \tilde{S}^n and \tilde{S} are both α -stable processes with the characteristic exponent ψ .

By (ii) and (4.44), we have

$$\tilde{X}_t^n = x + \tilde{S}_t^n + \int_0^t b^{(n)}(s+u, \tilde{X}_u^n) du \quad \tilde{\mathbf{P}}\text{-a.s.}, \quad \forall t \geq s. \tag{4.54}$$

According to (iii) and [5, Chap. 3, Lemma 7.7], there exists a countable set $I \subset \mathbb{R}_+$ such that

$$\tilde{\mathbf{P}}(\tilde{Z}_{t-} = \tilde{Z}_t) = 1, \quad \forall t \in \mathbb{R}_+ \setminus I. \tag{4.55}$$

It follows from (i), (4.55) and [5, Chap. 3, Prop. 5.2] that

$$\lim_{n \rightarrow \infty} \tilde{X}_t^n = \tilde{X}_t \quad \text{and} \quad \lim_{n \rightarrow \infty} \tilde{S}_t^n = \tilde{S}_t \quad \tilde{\mathbf{P}}\text{-a.s.}, \quad \forall t \in \mathbb{R}_+ \setminus I. \tag{4.56}$$

“Step 2”: Next, we show that, for any $f \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ and $\lambda > \lambda_0$,

$$\tilde{\mathbf{E}}\left[\int_0^\infty \exp(-\lambda t) f(t+s, \tilde{X}_t) dt\right] = G^\lambda f(s, x),$$

where $\tilde{\mathbf{E}}[\cdot]$ denotes the expectation taken with respect to the probability measure $\tilde{\mathbf{P}}$ on $(\tilde{\Omega}, \tilde{\mathcal{A}})$ and G^λ is defined in (4.25). We first consider $f \in C_b(\mathbb{R}_+ \times \mathbb{R}^d)$. By dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \tilde{\mathbf{E}}[f(t+s, \tilde{X}_t^n)] = \tilde{\mathbf{E}}[f(t+s, \tilde{X}_t)], \quad \forall t \in \mathbb{R}_+ \setminus I.$$

Since I is countable, by dominated convergence and Fubini's theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{\mathbf{E}} \left[\int_0^\infty \exp(-\lambda t) f(t+s, \tilde{X}_t^n) dt \right] &= \lim_{n \rightarrow \infty} \int_0^\infty \exp(-\lambda t) \tilde{\mathbf{E}}[f(t+s, \tilde{X}_t^n)] dt \\ &= \int_0^\infty \exp(-\lambda t) \tilde{\mathbf{E}}[f(t+s, \tilde{X}_t)] dt \\ &= \tilde{\mathbf{E}} \left[\int_0^\infty \exp(-\lambda t) f(t+s, \tilde{X}_t) dt \right]. \end{aligned} \quad (4.57)$$

From Lemma 4.3 we know that, for any $\lambda > \lambda_0$,

$$\tilde{\mathbf{E}} \left[\int_0^\infty \exp(-\lambda t) f(t+s, \tilde{X}_t^n) dt \right] = G_n^\lambda f(s, x). \quad (4.58)$$

Since $G_n^\lambda f \rightarrow G^\lambda f$ locally uniformly as $n \rightarrow \infty$ by Lemma 4.4, it follows from (4.57) and (4.58) that

$$\tilde{\mathbf{E}} \left[\int_0^\infty \exp(-\lambda t) f(t+s, \tilde{X}_t) dt \right] = G^\lambda f(s, x), \quad f \in C_b(\mathbb{R}_+ \times \mathbb{R}^d). \quad (4.59)$$

For any open subset O of $\mathbb{R}_+ \times \mathbb{R}^d$, we can find $f_n \in C_b(\mathbb{R}_+ \times \mathbb{R}^d)$, $n \in \mathbb{N}$, such that $0 \leq f_n \uparrow \mathbf{1}_O$ as $n \rightarrow \infty$. For $\lambda > \lambda_0$,

$$\begin{aligned} |G^\lambda \mathbf{1}_O(s, x) - G^\lambda f_n(s, x)| &= |G^\lambda (\mathbf{1}_O - f_n)(s, x)| \\ &\leq \sum_{k=0}^{\infty} |R^\lambda (BR^\lambda)^k (\mathbf{1}_O - f_n)|(s, x). \end{aligned} \quad (4.60)$$

For each $k \in \mathbb{Z}_+$, we have

$$\lim_{n \rightarrow \infty} |R^\lambda (BR^\lambda)^k (\mathbf{1}_O - f_n)|(s, x) = 0. \quad (4.61)$$

This can be achieved by applying dominated convergence theorem for $k+1$ times. By (4.60), (4.27) and (4.61), we see that

$$\lim_{n \rightarrow \infty} |G^\lambda (\mathbf{1}_O - f_n)|(s, x) = 0. \quad (4.62)$$

Since (4.58) is true for $f = f_n$, with $n \rightarrow \infty$, it follows from monotone convergence theorem that (4.59) is also true for $f = \mathbf{1}_O$. Now, we can use a monotone class argument to extend (4.59) to all $f \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$.

Similarly to (4.62), we know that $G^\lambda(|b| \wedge k)(s, x)$ goes to $G^\lambda(|b|)(s, x)$ as $k \rightarrow \infty$. By Remark 4.5 and monotone convergence theorem,

$$\begin{aligned} \tilde{\mathbf{E}} \left[\int_0^\infty \exp(-\lambda t) |b(t+s, \tilde{X}_t)| dt \right] &= \lim_{k \rightarrow \infty} \tilde{\mathbf{E}} \left[\int_0^\infty \exp(-\lambda t) (|b(t+s, \tilde{X}_t)| \wedge k) dt \right] \\ &= \lim_{k \rightarrow \infty} G^\lambda(|b| \wedge k)(s, x) \\ &= G^\lambda(|b|)(s, x) \leq \|G^\lambda(|b|)\| < \infty. \end{aligned} \quad (4.63)$$

Therefore, for each $t \geq 0$, we have $\int_0^t |b(t+s, \tilde{X}_u)| du < \infty$ $\tilde{\mathbf{P}}$ -a.s..

"Step 3": We next show that

$$\tilde{X}_t = x + \tilde{S}_t + \int_0^t b(s+u, \tilde{X}_u) du \quad \tilde{\mathbf{P}}\text{-a.s.}, \quad \forall t \in \mathbb{R}_+ \setminus I. \quad (4.64)$$

In view of (4.54) and (4.56), it suffices to show that, for each $t \geq 0$,

$$\int_0^t b^{(n)}(s+u, \tilde{X}_u^n) du \rightarrow \int_0^t b(s+u, \tilde{X}_u) du \quad \text{in probability as } n \rightarrow \infty. \quad (4.65)$$

For any $\delta > 0$ and $\lambda > \lambda_0$, by Chebyshev's inequality,

$$\begin{aligned} & \tilde{\mathbf{P}}\left(\left|\int_0^t b^{(n)}(s+u, \tilde{X}_u^n) du - \int_0^t b(s+u, \tilde{X}_u) du\right| > 3\delta\right) \\ & \leq \tilde{\mathbf{P}}\left(\left|\int_0^t b^{(k)}(s+u, \tilde{X}_u^n) du - \int_0^t b^{(k)}(s+u, \tilde{X}_u) du\right| > \delta\right) \\ & \quad + \tilde{\mathbf{P}}\left(\left|\int_0^t (b^{(n)} - b^{(k)})(s+u, \tilde{X}_u^n) du\right| > \delta\right) + \tilde{\mathbf{P}}\left(\left|\int_0^t (b - b^{(k)})(s+u, \tilde{X}_u) du\right| > \delta\right) \\ & \leq \delta^{-1} \tilde{\mathbf{E}}\left[\left|\int_0^t b^{(k)}(s+u, \tilde{X}_u^n) du - \int_0^t b^{(k)}(s+u, \tilde{X}_u) du\right|\right] \\ & \quad + \delta^{-1} \tilde{\mathbf{E}}\left[\int_0^t |b^{(n)} - b^{(k)}|(s+u, \tilde{X}_u^n) du\right] + \delta^{-1} \tilde{\mathbf{E}}\left[\int_0^t |b - b^{(k)}|(s+u, \tilde{X}_u) du\right] \\ & \leq \delta^{-1} e^{\lambda t} \left(e^{-\lambda t} \tilde{\mathbf{E}}\left[\int_0^t |b^{(k)}(s+u, \tilde{X}_u^n) - b^{(k)}(s+u, \tilde{X}_u)| du\right] \right. \\ & \quad \left. + \tilde{\mathbf{E}}\left[\int_0^t e^{-\lambda u} |b^{(n)} - b^{(k)}|(s+u, \tilde{X}_u^n) du\right] + \tilde{\mathbf{E}}\left[\int_0^t e^{-\lambda u} |b - b^{(k)}|(s+u, \tilde{X}_u) du\right] \right) \\ & \leq \delta^{-1} \int_0^t \tilde{\mathbf{E}}[|b^{(k)}(s+u, \tilde{X}_u^n) - b^{(k)}(s+u, \tilde{X}_u)|] du + e^{\lambda t} \delta^{-1} G_n^\lambda(|b^{(n)} - b^{(k)}|)(s, x) \\ & \quad + e^{\lambda t} \delta^{-1} G^\lambda(|b - b^{(k)}|)(s, x), \end{aligned} \quad (4.66)$$

where $k \in \mathbb{N}$ will be determined later. At the moment, we assume the following claim is true.

Claim 3. $\lim_{n, k \rightarrow \infty} G_n^\lambda(|b^{(n)} - b^{(k)}|)(s, x) = \lim_{k \rightarrow \infty} G^\lambda(|b - b^{(k)}|)(s, x) = 0$.

The proof of this claim will be given in “Step 4”. According to Claim 3, for any $\epsilon > 0$, we can choose k_0 large enough such that, for any $n \geq k_0$,

$$G^\lambda(|b - b^{(k_0)}|)(s, x) \leq \epsilon \delta e^{-\lambda t} / 3 \quad \text{and} \quad G_n^\lambda(|b^{(n)} - b^{(k_0)}|)(s, x) \leq \epsilon \delta e^{-\lambda t} / 3. \quad (4.67)$$

Noting that $b^{(k_0)}$ is bounded and globally Lipschitz in the space variable, by dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^t \tilde{\mathbf{E}}[|b^{(k_0)}(s+u, \tilde{X}_u^n) - b^{(k_0)}(s+u, \tilde{X}_u)|] du = 0. \quad (4.68)$$

Therefore, there exists $n_0 \geq k_0$ such that for $n \geq n_0$,

$$\int_0^t \tilde{\mathbf{E}}[|b^{(k_0)}(s+u, \tilde{X}_u^n) - b^{(k_0)}(s+u, \tilde{X}_u)|] du \leq \epsilon \delta / 3. \quad (4.69)$$

Combining (4.66), (4.67) and (4.69) yields

$$\tilde{\mathbf{P}}\left(\left|\int_0^t b^{(n)}(s+u, \tilde{X}_u^n) du - \int_0^t b(s+u, \tilde{X}_u) du\right| > 3\delta\right) \leq \epsilon, \quad \forall n \geq n_0,$$

which implies (4.65).

“Step 4”: In this step, we prove Claim 3. Let $\lambda > \lambda_0$. Since

$$|b^{(n)} - b^{(k)}| \leq |b_1^{(n)} - b_1^{(k)}| + |b_2^{(n)} - b_2^{(k)}|,$$

by (4.58), we have

$$G_n^\lambda(|b^{(n)} - b^{(k)}|) \leq G_n^\lambda(|b_1^{(n)} - b_1^{(k)}|) + G_n^\lambda(|b_2^{(n)} - b_2^{(k)}|). \quad (4.70)$$

It follows from Proposition 3.9 and Remark 4.2 that

$$\|\nabla R^\lambda(|b_2^{(n)} - b_2^{(k)}|)\| \leq M_\lambda \|b_2^{(n)} - b_2^{(k)}\|_{L^q([0,T];L^p(\mathbb{R}^d))}.$$

By (4.23), we have, for each $i \in \mathbb{N}$,

$$\begin{aligned} & \|R^\lambda(B_n R^\lambda)^i(|b_2^{(n)} - b_2^{(k)}|)\| \\ & \leq M_\lambda \|b_2^{(n)} - b_2^{(k)}\|_{L^q([0,T];L^p(\mathbb{R}^d))} (\kappa_\lambda)^{i-1} (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}). \end{aligned}$$

By Proposition 3.9 and (4.8),

$$\begin{aligned} & \|G_n^\lambda(|b_2^{(n)} - b_2^{(k)}|)\| \\ & \leq \|R^\lambda(|b_2^{(n)} - b_2^{(k)}|)\| + \sum_{i=1}^{\infty} \|R^\lambda(B_n R^\lambda)^i(|b_2^{(n)} - b_2^{(k)}|)\| \\ & \leq N_\lambda \|b_2^{(n)} - b_2^{(k)}\|_{L^q([0,T];L^p(\mathbb{R}^d))} + C_1 M_\lambda \|b_2^{(n)} - b_2^{(k)}\|_{L^q([0,T];L^p(\mathbb{R}^d))}, \end{aligned}$$

where $C_1 := (1 - \kappa_\lambda)^{-1} (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))})$ is a constant. It follows that $G_n^\lambda(|b_2^{(n)} - b_2^{(k)}|)$ converges uniformly to 0 as $n, k \rightarrow \infty$.

By (4.70), to show that $G_n^\lambda(|b^{(n)} - b^{(k)}|)(s, x)$ goes to 0 as $n, k \rightarrow \infty$, it suffices to show that

$$G_n^\lambda(|b_1^{(n)} - b_1^{(k)}|) \rightarrow 0 \quad \text{locally uniformly as } n, k \rightarrow \infty. \quad (4.71)$$

By (4.21), (4.23) and Lemma 3.10, we have, for each $i \in \mathbb{Z}_+$,

$$\|R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)\| \leq 2L_\lambda M (\kappa_\lambda)^{i-1} (M\lambda^{-1} + N_\lambda \|b_2\|_{L^q([0,T];L^p(\mathbb{R}^d))}) \quad (4.72)$$

and

$$\|\nabla R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)\| \leq 2L_\lambda M (\kappa_\lambda)^i. \quad (4.73)$$

Next, we show by induction that, for any $i \in \mathbb{Z}_+$ and compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$,

$$\lim_{n,k \rightarrow \infty} \sup_{(t,y) \in K} R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)(t, y) = 0 \quad (4.74)$$

and

$$\lim_{n,k \rightarrow \infty} \sup_{(t,y) \in K} |\nabla R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)(t, y)| = 0 \quad (4.75)$$

For $i = 0$, by (4.39), it is easy to see that

$$\lim_{n,k \rightarrow \infty} \sup_{(t,y) \in K} R^\lambda(|b_1^{(n)} - b_1^{(k)}|)(t, y) = 0.$$

Very similarly, we also have

$$\lim_{n,k \rightarrow \infty} \sup_{(t,y) \in K} |\nabla R^\lambda(|b_1^{(n)} - b_1^{(k)}|)(t, y)| = 0.$$

Suppose that (4.74) and (4.75) are true for i . For $m > 0$ let A_m and h_m be as in (4.30) and (4.31), respectively. Set

$$C_{n,k,m} := \sup_{(t,y) \in [0,T] \times A_m} |\nabla R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)| (t, y).$$

By induction hypothesis, $\lim_{n,k \rightarrow \infty} C_{n,k,m} = 0$ for any $m > 0$. It follows from (4.73) that

$$\begin{aligned} & |R^\lambda(B_n R^\lambda)^{i+1}(|b_1^{(n)} - b_1^{(k)}|)| (t, y) \\ &= |R^\lambda(b^{(n)}(h_m + 1 - h_m) \cdot \nabla R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)| (t, y) \\ &\leq |R^\lambda(b^{(n)} h_m \cdot \nabla R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)| (t, y) \\ &\quad + |R^\lambda(b^{(n)}(1 - h_m) \cdot \nabla R^\lambda(B_n R^\lambda)^i(|b_1^{(n)} - b_1^{(k)}|)| (t, y) \\ &\leq C_{n,k,m} \|R^\lambda(|b^{(n)}|)\| + 2L_\lambda M(\kappa_\lambda)^i |R^\lambda(|b^{(n)}|(1 - h_m))| (t, y). \end{aligned} \quad (4.76)$$

Combining (4.33), (4.41) and (4.76), we obtain (4.74) for $i + 1$. The claim (4.75) for $i + 1$ can be similarly proved. Therefore, (4.74) and (4.75) are true for any $i \in \mathbb{Z}_+$.

By (4.24), (4.72) and (4.74), we see that (4.71) holds. As a consequence,

$$\lim_{n,k \rightarrow \infty} G_n^\lambda(|b^{(n)} - b^{(k)}|)(s, x) = 0.$$

With a very similar argument as above, we conclude that

$$\lim_{k \rightarrow \infty} G^\lambda(|b - b^{(k)}|)(s, x) = 0.$$

Thus Claim 3 is proved.

“Step 5”: Since I is countable and \tilde{X} , \tilde{S} and $\int_0^t b(u + s, \tilde{X}_u) du$ all have càdlàg paths, (4.64) must hold for all $t \geq 0$. This completes the proof. \square

Corollary 4.8. *Assume the same assumptions as in Theorem 4.7. For each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, let $X^{s,x} = (X_t^{s,x})_{t \geq 0}$ be the solution of (1.1) that we constructed in Theorem 4.7. Define the process $Y^{s,x} = (Y_t^{s,x})_{t \geq 0}$ by*

$$Y_t^{s,x} = \begin{cases} x, & 0 \leq t \leq s, \\ X_{t-s}^{s,x}, & t > s. \end{cases}$$

Let $\mathbf{P}^{s,x}$ be the probability measure on the path space (D, \mathcal{D}) induced by $Y^{s,x}$. Then $\mathbf{P}^{s,x}$ is a solution to the martingale problem for $L_t = A + b(t, \cdot) \cdot \nabla$ starting from (s, x) . Moreover, the family of measures $\{\mathbf{P}^{s,x} : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is measurable, that is, $\mathbf{P}^{s,x}(A)$ is measurable in (s, x) for every $A \in \mathcal{D}$.

Proof. A simple application of Itô’s formula leads to the fact that $\mathbf{P}^{s,x}$ is a solution to the martingale problem for L_t starting from (s, x) .

Let $X = (X_t)_{t \geq 0}$ be the canonical process on (D, \mathcal{D}) . Let $\mathbf{E}^{s,x}[\cdot]$ denote the expectation taken with respect to the measure $\mathbf{P}^{s,x}$ on (D, \mathcal{D}) . It follows from (4.43) that, for any $\lambda > \lambda_0$ and $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\mathbf{E}^{s,x} \left[\int_s^\infty e^{-\lambda(t-s)} g(t, X_t) dt \right] = G^\lambda g(s, x) = \sum_{i=0}^\infty R^\lambda (B R^\lambda)^i g(s, x). \quad (4.77)$$

We next show that the family $\{\mathbf{P}^{s,x} : (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is measurable. Let $\varphi \in C_b(\mathbb{R}^d)$ and $t > 0$. Define $\tilde{\varphi}_n : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ as follows:

$$\tilde{\varphi}_n(u, y) := \begin{cases} 0, & u < t, \\ \varphi(y)\rho_n(u-t), & u \geq t, \end{cases}$$

where ρ_n is a mollifying sequence on \mathbb{R} with $\rho_n(u) = \rho_n(-u)$, $u \in \mathbb{R}$. By (4.77), for $s \in [0, t)$, $x \in \mathbb{R}^d$ and $\lambda > \lambda_0$,

$$\mathbf{E}^{s,x} \left[\int_s^\infty e^{-\lambda(u-s)} \tilde{\varphi}_n(u, X_u) du \right] = \sum_{i=0}^\infty R^\lambda (BR^\lambda)^i \tilde{\varphi}_n(s, x).$$

It follows that $\mathbf{E}^{s,x}[\int_s^\infty e^{-\lambda(u-s)} \tilde{\varphi}_n(u, X_u) du]$ is measurable in (s, x) . By dominated convergence theorem and noting that X_t is right-continuous, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{E}^{s,x} \left[\int_s^\infty e^{-\lambda(u-s)} \tilde{\varphi}_n(u, X_u) du \right] &= \lim_{n \rightarrow \infty} \mathbf{E}^{s,x} \left[\int_t^\infty e^{-\lambda(u-s)} \varphi(X_u) \rho_n(u-t) du \right] \\ &= \mathbf{E}^{s,x} [2^{-1} e^{-\lambda(t-s)} \varphi(X_t)] \end{aligned}$$

for all $(s, x) \in [0, t) \times \mathbb{R}^d$, which implies that $\mathbf{E}^{s,x}[\varphi(X_t)]$ is measurable in $(s, x) \in [0, t) \times \mathbb{R}^d$. If $s \geq t$, then $\mathbf{E}^{s,x}[\varphi(X_t)] = \varphi(x)$. Thus $\mathbf{E}^{s,x}[\varphi(X_t)]$ is measurable in $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Similarly, for $0 \leq r_1 \leq \dots \leq r_l$ and $g_1, \dots, g_l \in C_b(\mathbb{R}^d)$ with $l \in \mathbb{N}$, one can show that $\mathbf{E}^{s,x}[\prod_{j=1}^l g_j(X_{r_j})]$ is measurable in $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Now, the assertion follows by a monotone class argument. \square

The following lemma is analog to [20, Theorem 6.1.3], which plays an important role in showing the uniqueness of solutions to martingale problems. With this lemma, we can use the standard argument to show that multi-dimensional distributions of solutions to the martingale problem for L_t are unique, provided that one-dimensional distributions of those are so. Recall that $X = (X_t)_{t \geq 0}$ is the canonical process defined on the path space (D, \mathcal{D}) and $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by $(X_t)_{t \geq 0}$.

Lemma 4.9. *Suppose that the probability measure $\mathbf{Q}^{s,x}$ on $(D = D([0, \infty); \mathbb{R}^d), \mathcal{D})$ satisfies:*

(i) $\mathbf{Q}^{s,x}$ is a solution to the martingale problem for L_t starting from (s, x) , where $L_t = A + b(t, \cdot) \cdot \nabla$;

(ii) $\mathbf{E}_{\mathbf{Q}^{s,x}} \left[\int_s^\infty e^{-\lambda(t-s)} |b(t, X_t)| dt \right] < \infty$.

For a given $t \geq s$, we denote by $Q_\omega(A) = Q(\omega, A) : \Omega \times \mathcal{F}_t \rightarrow [0, 1]$ the regular conditional distribution of $\mathbf{Q}^{s,x}$ given \mathcal{F}_t . Then there exists a $\mathbf{Q}^{s,x}$ -null set $N \in \mathcal{F}_t$ such that, for each $\omega \notin N$, Q_ω solves the martingale problem for L_t starting from $(t, \omega(t))$ and

$$\mathbf{E}_{Q_\omega} \left[\int_t^\infty e^{-\lambda(u-s)} |b(u, X_u)| du \right] < \infty.$$

Proof. We follow the proof of [20, Theorem 6.1.3]. Let $\{f_n : f_n \in C_0^\infty(\mathbb{R}^d), n \in \mathbb{N}\}$ be dense in $C_0^\infty(\mathbb{R}^d)$. By [20, Theorem 1.2.10], for each f_n , there exists $N_n \in \mathcal{F}_t$

such that $\mathbf{Q}^{s,x}(N_n) = 0$ and, for all $\omega \notin N_n$,

$$M_u^{f_n} := f_n(X_u) - f_n(X_t) - \int_t^u L_r f_n(r, X_r) dr, \quad u \geq t,$$

is an \mathcal{F}_u -martingale after time t with respect to Q_ω .

By (ii), we have

$$\int_D \mathbf{E}_{Q_\omega} \left[\int_t^\infty e^{-\lambda(u-s)} |b(u, X_u)| du \right] \mathbf{Q}^{s,x}(d\omega) < \infty,$$

so there exists $\mathbf{Q}^{s,x}$ -null set $N_0 \in \mathcal{F}_t$ such that

$$\mathbf{E}_{Q_\omega} \left[\int_t^\infty e^{-\lambda(u-s)} |b(u, X_u)| du \right] < \infty \quad \text{for all } \omega \notin N_0. \quad (4.78)$$

Let $N := \cup_{n \geq 0} N_n$.

We now fix $\omega \in \Omega \setminus N$. For any $f \in C_0^\infty(\mathbb{R}^d)$, we can find f_{n_k} such that $f_{n_k} \rightarrow f$ in $C_0^\infty(\mathbb{R}^d)$ as $k \rightarrow \infty$. In view of (4.78), we have for all $u \geq t$,

$$M_u^{f_{n_k}} \rightarrow M_u^f \quad Q_\omega\text{-a.s.} \quad \text{as } k \rightarrow \infty.$$

By (4.78), dominated convergence theorem and the martingale property of $M^{f_{n_k}}$, we see that M^f is also an \mathcal{F}_u -martingale after t with respect to Q_ω . Thus Q_ω solves the martingale problem for L_t starting from $(t, \omega(t))$. \square

Proposition 4.10. *Let $d \geq 2$ and $1 < \alpha < 2$. Assume that the α -stable process S is non-degenerate and Assumption 4.1 is satisfied. Then for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, solutions of the SDE (1.1) are weakly unique.*

Proof. Our proof is adapted from the proof of [1, Proposition 5.1]. Consider an arbitrary weak solution of (1.1), that is, a non-degenerate α -stable process $S = (S_t)_{t \geq 0}$ and a càdlàg process $Y = (Y_t)_{t \geq 0}$ that are both defined on some probability space $(\Omega, \mathcal{A}, \mathbf{Q})$ and are such that

$$Y_t = x + S_t + \int_0^t b(s+u, Y_u) du \quad \text{a.s.,} \quad \forall t \geq 0.$$

Define a measurable map $\Phi : \Omega \rightarrow D$ by

$$\Omega \ni \omega \mapsto \Phi(\omega) \in D, \quad \text{where} \quad \Phi(\omega)(t) := \begin{cases} x, & t \leq s, \\ Y_{t-s}(\omega), & t > s. \end{cases}$$

Let $\mathbf{Q}^{s,x}$ be the image measure of \mathbf{Q} on (D, \mathcal{D}) under the map Φ . Then it is routine to check that $\mathbf{Q}^{s,x}$ is a solution to the martingale problem for $L_t = A + b(t, \cdot) \cdot \nabla$ starting from (s, x) . To show that weak uniqueness for (1.1) holds, it suffices to prove $\mathbf{Q}^{s,x} = \mathbf{P}^{s,x}$ on (D, \mathcal{D}) , where $\mathbf{P}^{s,x}$ is defined in Corollary 4.8.

Let $(\mathcal{M}_t)_{t \geq 0}$ be the usual augmentation of $(\mathcal{F}_t)_{t \geq 0}$ with respect to the measure $\mathbf{Q}^{s,x}$. Define a sequence of \mathcal{F}_{t+} -stopping times

$$\sigma_n := \inf\{t \geq s : \int_s^t |b(u, X_u)| du > n\}, \quad n \in \mathbb{N},$$

and let

$$\tau_n := \sigma_n \wedge n \quad \text{for } n \in \mathbb{N} \quad \text{with } n \geq s.$$

Clearly, σ_n and τ_n are also \mathcal{M}_t -stopping times. According to the condition (1.3), we have $\tau_n \rightarrow \infty$ $\mathbf{Q}^{s,x}$ -a.s..

For each fixed $\omega \in D$, it follows from [20, Lemma 6.1.1] that there is a unique probability measure $\delta_\omega \otimes_{\tau_n(\omega)} \mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}$ on (D, \mathcal{D}) such that

$$\delta_\omega \otimes_{\tau_n(\omega)} \mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}(X_t = \omega(t), 0 \leq t \leq \tau_n(\omega)) = 1$$

and

$$\delta_\omega \otimes_{\tau_n(\omega)} \mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}(A) = \mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}(A), \quad A \in \mathcal{F}^{\tau_n(\omega)},$$

where $\mathcal{F}^t := \sigma(X(r) : r \geq t)$ for $t \geq 0$. In view of Corollary 4.8, we can easily check that $\delta_{(\cdot)} \otimes_{\tau_n(\cdot)} \mathbf{P}^{\tau_n(\cdot), (\cdot)(\tau_n)}$ is a probability kernel from $(D, \mathcal{M}_{\tau_n})$ to (D, \mathcal{D}) . For details, the reader is referred to the proof of [20, Theorem 6.1.2]. Thus $\delta_{(\cdot)} \otimes_{\tau_n(\cdot)} \mathbf{P}^{\tau_n(\cdot), (\cdot)(\tau_n)}$ induces a probability measure $\mathbf{Q}_n^{s,x}$ on (D, \mathcal{D}) with

$$\mathbf{Q}_n^{s,x}(A) = \int_D \delta_\omega \otimes_{\tau_n(\omega)} \mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}(A) \mathbf{Q}_n^{s,x}(d\omega), \quad A \in \mathcal{D}.$$

For each $\lambda > \lambda_0$,

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\int_s^\infty e^{-\lambda(t-s)} |b(t, X_t)| dt \right] \\ &= \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\int_s^{\tau_n} e^{-\lambda(t-s)} |b(t, X_t)| dt \right] \\ & \quad + \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[e^{-\lambda(\tau_n-s)} \mathbf{E}_{\mathbf{P}^{\tau_n, X_{\tau_n}}} \left[\int_{\tau_n}^\infty e^{-\lambda(t-\tau_n)} |b(t, X_t)| dt \right] \right] \\ & \leq n + \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[e^{-\lambda(\tau_n-s)} \mathbf{E}_{\mathbf{P}^{\tau_n, X_{\tau_n}}} \left[\int_{\tau_n}^\infty e^{-\lambda(t-\tau_n)} |b(t, X_t)| dt \right] \right]. \end{aligned} \quad (4.79)$$

Just as in (4.63), we have

$$\mathbf{E}_{\mathbf{P}^{\tau_n, X_{\tau_n}}} \left[\int_{\tau_n}^\infty e^{-\lambda(t-\tau_n)} |b(t, X_t)| dt \right] = G^\lambda(|b|)(\tau_n, X_{\tau_n}) \leq \|G^\lambda(|b|)\| < \infty. \quad (4.80)$$

It follows from (4.79) and (4.80) that

$$\mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\int_s^\infty e^{-\lambda(t-s)} |b(t, X_t)| dt \right] < \infty. \quad (4.81)$$

Next, we proceed to show $\mathbf{Q}_n^{s,x} = \mathbf{P}^{s,x}$. For $f \in C_0^\infty(\mathbb{R}^d)$, set

$$M_t^f := f(X_t) - f(X_s) - \int_s^t L_u f(u, X_u) du, \quad t \geq s.$$

Then $M^f := (M_t^f)_{t \geq s}$ is an \mathcal{F}_t -martingale after time s with respect to the measure $\mathbf{Q}_n^{s,x}$. To see this, let $s \leq t_1 \leq t_2$ and $A \in \mathcal{F}_{t_1}$. Then

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}_n^{s,x}}[M_{t_2}^f; A] \\ &= \int_{\{\tau_n \leq t_1\}} \mathbf{E}_{\mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}}[M_{t_2}^f; A] \mathbf{Q}^{s,x}(d\omega) + \int_{\{t_1 < \tau_n \leq t_2\}} M_{\tau_n}^f(\omega) \mathbf{1}_A(\omega) \mathbf{Q}^{s,x}(d\omega) \\ & \quad + \int_{\{\tau_n > t_2\}} M_{t_2}^f(\omega) \mathbf{1}_A(\omega) \mathbf{Q}^{s,x}(d\omega) \\ &= \int_{\{\tau_n \leq t_1\}} \mathbf{E}_{\mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}}[M_{t_1}^f; A] \mathbf{Q}^{s,x}(d\omega) + \mathbf{E}_{\mathbf{Q}^{s,x}}[M_{t_2 \wedge \tau_n}^f; A \cap \{\tau_n > t_1\}]. \end{aligned}$$

Since $A \cap \{\tau_n > t_1\} \in \mathcal{M}_{t_1 \wedge \tau_n}$, by optional sampling theorem,

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}^{s,x}}[M_{t_2 \wedge \tau_n}^f; A \cap \{\tau_n > t_1\}] &= \mathbf{E}_{\mathbf{Q}^{s,x}}[M_{t_1 \wedge \tau_n}^f; A \cap \{\tau_n > t_1\}] \\ &= \mathbf{E}_{\mathbf{Q}^{s,x}}[M_{t_1}^f; A \cap \{\tau_n > t_1\}]. \end{aligned}$$

So

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}_n^{s,x}}[M_{t_2}^f; A] \\ &= \int_{\{\tau_n \leq t_1\}} \mathbf{E}_{\mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}}[M_{t_1}^f; A] \mathbf{Q}^{s,x}(d\omega) + \mathbf{E}_{\mathbf{Q}^{s,x}}[M_{t_1}^f; A \cap \{\tau_n > t_1\}] \\ &= \int_{\{\tau_n \leq t_1\}} \mathbf{E}_{\mathbf{P}^{\tau_n(\omega), \omega(\tau_n)}}[M_{t_1}^f; A] \mathbf{Q}^{s,x}(d\omega) + \int_{\{\tau_n > t_1\}} M_{t_1}^f(\omega) \mathbf{1}_A(\omega) \mathbf{Q}^{s,x}(d\omega) \\ &= \mathbf{E}_{\mathbf{Q}_n^{s,x}}[M_{t_1}^f; A]. \end{aligned}$$

This shows that M^f is an \mathcal{F}_t -martingale after time s with respect to $\mathbf{Q}_n^{s,x}$. It follows from (4.81) and [20, Theorem 4.2.1] that, for any $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$f(t, X_t) - f(s, X_s) - \int_s^t \left(\frac{\partial f}{\partial u} + L_u f \right)(u, X_u) du$$

is an \mathcal{F}_t -martingale after time s with respect to $\mathbf{Q}_n^{s,x}$. As a consequence,

$$\mathbf{E}_{\mathbf{Q}_n^{s,x}}[f(t, X_t)] - f(s, x) = \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\int_s^t \left(\frac{\partial f}{\partial u} + L_u f \right)(u, X_u) du \right].$$

We now define a linear functional V_n^λ as follows. For any measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d$ with

$$\mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\int_s^\infty e^{-\lambda(t-s)} |f(t, X_t)| dt \right] < \infty,$$

let

$$V_n^\lambda f := \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\int_s^\infty e^{-\lambda(t-s)} f(t, X_t) dt \right].$$

By the same argument that we used to obtain (4.18), we get

$$V_n^\lambda g = R^\lambda g(s, x) + V_n^\lambda B R^\lambda g, \quad g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d). \quad (4.82)$$

where BR^λ is defined by (4.9). Since, by (4.81), $V_n^\lambda(|b|) < \infty$, we can use the equality (4.82) and dominated convergence theorem to get

$$V_n^\lambda BR^\lambda g = R^\lambda BR^\lambda g(s, x) + V_n^\lambda (BR^\lambda)^2 g, \quad g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d),$$

which implies

$$V_n^\lambda g = R^\lambda g(s, x) + R^\lambda BR^\lambda g(s, x) + V_n^\lambda (BR^\lambda)^2 g, \quad g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d).$$

After a simple induction, we obtain

$$V_n^\lambda g = \sum_{i=0}^k R^\lambda (BR^\lambda)^i g(s, x) + V_n^\lambda (BR^\lambda)^{k+1} g, \quad g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d).$$

For $\lambda > \lambda_0$, by (4.12) and (4.26), we get

$$|V_n^\lambda (BR^\lambda)^{k+1} g| \leq \|\nabla R^\lambda (BR^\lambda)^k g\| V_n^\lambda(|b|) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It follows from Lemma 4.4 and (4.77) that, for any $\lambda > \lambda_0$ and $g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\int_s^\infty e^{-\lambda(t-s)} g(t, X_t) dt \right] &= V_n^\lambda g = \sum_{i=0}^\infty R^\lambda (BR^\lambda)^i g(s, x) \\ &= \mathbf{E}_{\mathbf{P}^{s,x}} \left[\int_s^\infty e^{-\lambda(t-s)} g(t, X_t) dt \right]. \end{aligned}$$

By the uniqueness of the Laplace transform, we have

$$\mathbf{E}_{\mathbf{Q}_n^{s,x}}[f(X_t)] = \mathbf{E}_{\mathbf{P}^{s,x}}[f(X_t)], \quad \forall f \in C_b(\mathbb{R}^d), \quad t \geq s.$$

This means that one-dimensional distributions of $\mathbf{Q}_n^{s,x}$ and $\mathbf{P}^{s,x}$ are the same. By (4.81), Lemma 4.9 and the argument in the proof of [20, Theorem 6.2.3], we conclude that multiple-dimensional distributions of $\mathbf{Q}_n^{s,x}$ and $\mathbf{P}_n^{s,x}$ also coincide, that is, $\mathbf{Q}_n^{s,x} = \mathbf{P}_n^{s,x}$ on (D, \mathcal{D}) .

Note that $\tau_n \rightarrow \infty$ $\mathbf{Q}_n^{s,x}$ -a.s.. With the same reason, we have $\tau_n \rightarrow \infty$ $\mathbf{P}^{s,x}$ -a.s.. Since $\mathbf{Q}_n^{s,x}$ and $\mathbf{Q}_n^{s,x}$ coincide before τ_n , it follows from dominated convergence that, for $0 \leq r_1 \leq \dots \leq r_l$ and $g_1, \dots, g_l \in C_b(\mathbb{R}^d)$ with $l \in \mathbb{N}$,

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\prod_{j=1}^l g_j(X_{r_j}) \right] &= \lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\prod_{j=1}^l g_j(X_{r_j \wedge \tau_n}) \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[\prod_{j=1}^l g_j(X_{r_j \wedge \tau_n}) \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E}_{\mathbf{P}^{s,x}} \left[\prod_{j=1}^l g_j(X_{r_j \wedge \tau_n}) \right] \\ &= \mathbf{E}_{\mathbf{P}^{s,x}} \left[\prod_{j=1}^l g_j(X_{r_j}) \right]. \end{aligned}$$

Thus $\mathbf{Q}^{s,x} = \mathbf{P}^{s,x}$ on (D, \mathcal{D}) . This completes the proof. \square

5. Existence and Uniqueness of Weak Solutions: Global Case

In this section we study the general case and prove Theorem 1.1. In contrast to Theorem 4.7 and Proposition 4.10, the main task here is to remove the restriction that $\text{supp}(b) \subset [0, T] \times \mathbb{R}^d$, which was assumed in the previous section.

Proof of Theorem 1.1. “*Existence*”: For each $n \in \mathbb{N}$, consider the drift $a_n : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by

$$a_n(t, \cdot) := \begin{cases} b(t, \cdot), & t \leq n, \\ 0, & \text{otherwise.} \end{cases} \quad (5.1)$$

According to Theorem 4.7, there exist a càdlàg process $X^n = (X_t^n)_{t \geq 0}$ and a non-degenerate α -stable process $S^n = (S_t^n)_{t \geq 0}$ with characteristic exponent ψ that are both defined on some probability space $(\Omega_n, \mathcal{A}_n, \mathbf{P}_n)$ such that

$$X_t^n = x + S_t^n + \int_0^t a_n(s + u, X_u^n) du \quad \text{a.s.,} \quad \forall t \geq 0. \quad (5.2)$$

Define $\Phi_n : (\Omega_n, \mathcal{A}_n) \rightarrow (D, \mathcal{D})$ by

$$\Omega_n \ni \omega \mapsto \Phi_n(\omega) \in D, \quad \text{where} \quad \Phi_n(\omega)(t) := \begin{cases} x, & t \leq s, \\ X_{t-s}^n(\omega), & t > s. \end{cases}$$

Consider the measure $\mathbf{Q}_n^{s,x}$ on (D, \mathcal{D}) defined as $\mathbf{Q}_n^{s,x} := \mathbf{P}_n \circ (\Phi_n)^{-1}$, that is, $\mathbf{Q}_n^{s,x}$ is the image measure of \mathbf{P}_n under the map Φ_n . Since $\text{supp}(a_n) \subset [0, T] \times \mathbb{R}^d$, by the local weak uniqueness for (5.2) that we have shown in Proposition 4.10, the measures $\mathbf{Q}_n^{s,x}$, $n \in \mathbb{N}$, must be consistent, that is, $\mathbf{Q}_{n+1}^{s,x}|_{\mathcal{F}_n} = \mathbf{Q}_n^{s,x}|_{\mathcal{F}_n}$ for all $n \in \mathbb{N}$. It follows from the projective limit theorem (see, e.g., [10, Corollary 6.15]) that there exists a probability measure $\mathbf{Q}^{s,x}$ on (D, \mathcal{D}) such that $\mathbf{Q}^{s,x}|_{\mathcal{F}_n} = \mathbf{Q}_n^{s,x}|_{\mathcal{F}_n}$ for all $n \in \mathbb{N}$. Let $X = (X_t)_{t \geq 0}$ be the canonical process defined on (D, \mathcal{D}) . For any $\xi \in \mathbb{R}^d$ and $t \geq 0$, by choosing $n \in \mathbb{N}$ such that $n \geq t + s$, we obtain

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}^{s,x}} \left[e^{i\xi \cdot (X_{s+t} - x - \int_0^t b(s+u, X_{s+u}) du)} \right] &= \mathbf{E}_{\mathbf{Q}_n^{s,x}} \left[e^{i\xi \cdot (X_{s+t} - x - \int_0^t b(s+u, X_{s+u}) du)} \right] \\ &= \mathbf{E}_{\mathbf{P}_n} \left[e^{i\xi \cdot (X_t^n - x - \int_0^t b(s+u, X_u^n) du)} \right] \\ &= \mathbf{E}_{\mathbf{P}_n} \left[e^{i\xi \cdot (X_t^n - x - \int_0^t a_n(s+u, X_u^n) du)} \right] \\ &= \mathbf{E}_{\mathbf{P}_n} \left[e^{i\xi \cdot S_t^n} \right] = e^{-t\psi(\xi)}, \end{aligned}$$

that is, under the measure $\mathbf{Q}^{s,x}$, the process

$$S_t := X_{s+t} - x - \int_0^t b(s+u, X_{s+u}) du, \quad t \geq 0,$$

is an α -stable process with characteristic exponent ψ . Define $\tilde{X} := (\tilde{X}_t)_{t \geq 0}$ with $\tilde{X}_t := X_{t+s}$, $t \geq 0$. Then \tilde{X} satisfies

$$\tilde{X}_t = x + S_t + \int_0^t b(s+u, \tilde{X}_u) du \quad \mathbf{Q}^{s,x}\text{-a.s.,} \quad \forall t \geq 0.$$

Thus \tilde{X} is a weak solution to the SDE (1.1).

“*Uniqueness*”: Consider an arbitrary weak solution of (1.1), that is, a non-degenerate α -stable process $S = (S_t)_{t \geq 0}$ and a càdlàg process $Y = (Y_t)_{t \geq 0}$ that are both defined on some probability space $(\Omega, \mathcal{A}, \mathbf{Q})$ such that

$$Y_t = x + S_t + \int_0^t b(s+u, Y_u) du \quad \text{a.s.,} \quad \forall t \geq 0.$$

For $k \in \mathbb{N}$ with $k \geq s$, let a_k be as in (5.1), and define $Y_t^k := Y_t$ for $0 \leq t \leq k - s$ and

$$Y_t^k := Y_{k-s} + S_t - S_{k-s}, \quad t > k - s.$$

Thus

$$Y_t^k = x + S_t + \int_0^t a_k(s + u, Y_u^k) du \quad \text{a.s.,} \quad \forall t \geq 0.$$

By Proposition 4.10, the law of $Y^k := (Y_t^k)_{t \geq 0}$ is uniquely determined. Since $Y_t = Y_t^k$ for $t \leq k - s$, the law of the process Y is uniquely determined at least up to time $k - s$. With $k \rightarrow \infty$, we see that the law of X is completely and uniquely determined. \square

References

1. Richard F. Bass and Zhen-Qing Chen, *Brownian motion with singular drift*, Ann. Probab. **31** (2003), no. 2, 791–817.
2. Zhen-Qing Chen, Renming Song, and Xicheng Zhang, *Stochastic flows for Lévy processes with Hölder drifts*, arXiv preprint arXiv:1501.04758 (2015).
3. Zhen-Qing Chen and Longmin Wang, *Uniqueness of stable processes with drift*, arXiv preprint arXiv:1309.6414 (2013).
4. Kai Lai Chung, *Lectures from Markov processes to Brownian motion*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 249, Springer-Verlag, New York-Berlin, 1982.
5. Stewart N. Ethier and Thomas G. Kurtz, *Markov processes: characterization and convergence*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986.
6. F. Flandoli, M. Gubinelli, and E. Priola, *Well-posedness of the transport equation by stochastic perturbation*, Invent. Math. **180** (2010), no. 1, 1–53.
7. Sven Haadem and Frank Proske, *On the construction and Malliavin differentiability of solutions of Lévy noise driven SDE's with singular coefficients*, J. Funct. Anal. **266** (2014), no. 8, 5321–5359.
8. Nobuyuki Ikeda and Shinzo Watanabe, *Stochastic differential equations and diffusion processes*, second ed., North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.
9. Peng Jin, *Stochastic dynamics with singular lower order terms in finite and infinite dimensions*, PhD dissertation, University of Bielefeld, 2009.
10. Olav Kallenberg, *Foundations of modern probability*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002.
11. Panki Kim and Renming Song, *Stable process with singular drift*, Stochastic Process. Appl. **124** (2014), no. 7, 2479–2516.
12. Takashi Komatsu, *On the martingale problem for generators of stable processes with perturbations*, Osaka J. Math. **21** (1984), no. 1, 113–132.
13. N. V. Krylov and M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift*, Probab. Theory Related Fields **131** (2005), no. 2, 154–196.
14. V. P. Kurenok, *Stochastic equations with time-dependent drift driven by Lévy processes*, J. Theoret. Probab. **20** (2007), no. 4, 859–869.
15. S. I. Podolynny and N. I. Portenko, *On multidimensional stable processes with locally unbounded drift*, Random Oper. Stochastic Equations **3** (1995), no. 2, 113–124.
16. N. I. Portenko, *Generalized diffusion processes*, Translations of Mathematical Monographs, vol. 83, American Mathematical Society, Providence, RI, 1990, Translated from the Russian by H. H. McFaden.
17. Enrico Priola, *Pathwise uniqueness for singular SDEs driven by stable processes*, Osaka J. Math. **49** (2012), no. 2, 421–447.

18. Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999, Translated from the 1990 Japanese original, Revised by the author.
19. Rong Situ, *Theory of stochastic differential equations with jumps and applications*, Mathematical and Analytical Techniques with Applications to Engineering, Springer, New York, 2005, Mathematical and analytical techniques with applications to engineering.
20. Daniel W. Stroock and S. R. Srinivasa Varadhan, *Multidimensional diffusion processes*, Classics in Mathematics, Springer-Verlag, Berlin, 2006, Reprint of the 1997 edition.
21. Wolfgang Stummer, *The Novikov and entropy conditions of multidimensional diffusion processes with singular drift*, Probab. Theory Related Fields **97** (1993), no. 4, 515–542.
22. A. Ju. Veretennikov, *Strong solutions of stochastic differential equations*, Teor. Veroyatnost. i Primenen. **24** (1979), no. 2, 348–360.
23. Toshiro Watanabe, *Asymptotic estimates of multi-dimensional stable densities and their applications*, Trans. Amer. Math. Soc. **359** (2007), no. 6, 2851–2879 (electronic).
24. Xicheng Zhang, *Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients*, Electron. J. Probab. **16** (2011), no. 38, 1096–1116.
25. ———, *Stochastic differential equations with Sobolev drifts and driven by α -stable processes*, Ann. Inst. Henri Poincaré Probab. Stat. **49** (2013), no. 4, 1057–1079.
26. A. K. Zvonkin, *A transformation of the phase space of a diffusion process that will remove the drift*, Mat. Sb. (N.S.) **93(135)** (1974), 129–149, 152.

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